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Nonlinear hydrodynamic instability in eccentric astrophysical discs with vertical structure

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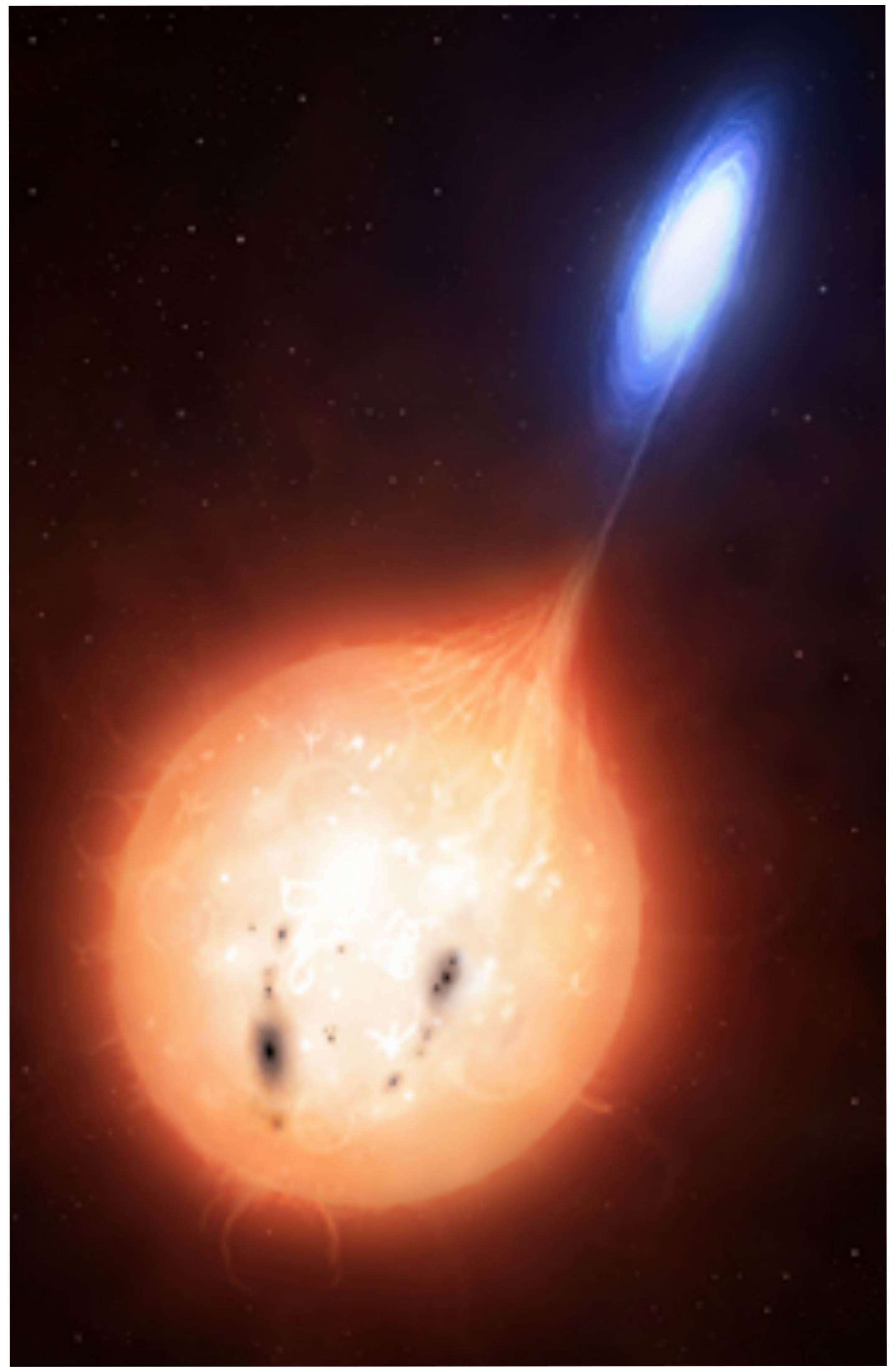
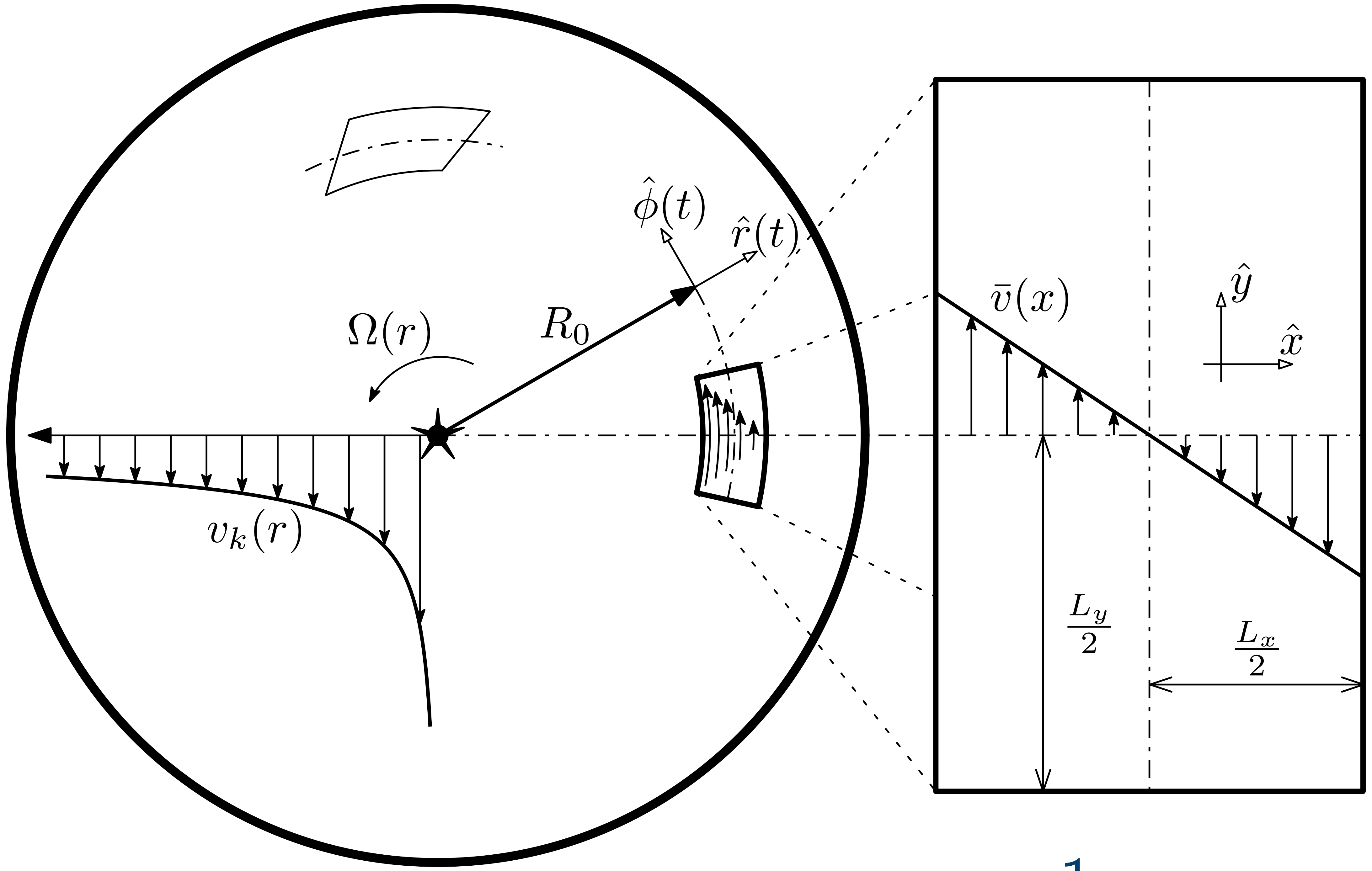
Motivation: Accretion Discs



Artistic Rendering by M. Garlick

- Classical theory assumes thin, circular discs (Pringle 1981)
- Eccentricity may be driven by
  - Anisotropic accretion onto the disc
  - Secular interactions with embedded planets
  - **Mean-motion (parametric) resonance with the tidal potential of a circular companion (Whitehurst 1988)**
- *How are these Lindblad resonances saturated?*
- *What dynamics are excited inside an eccentric disc?*

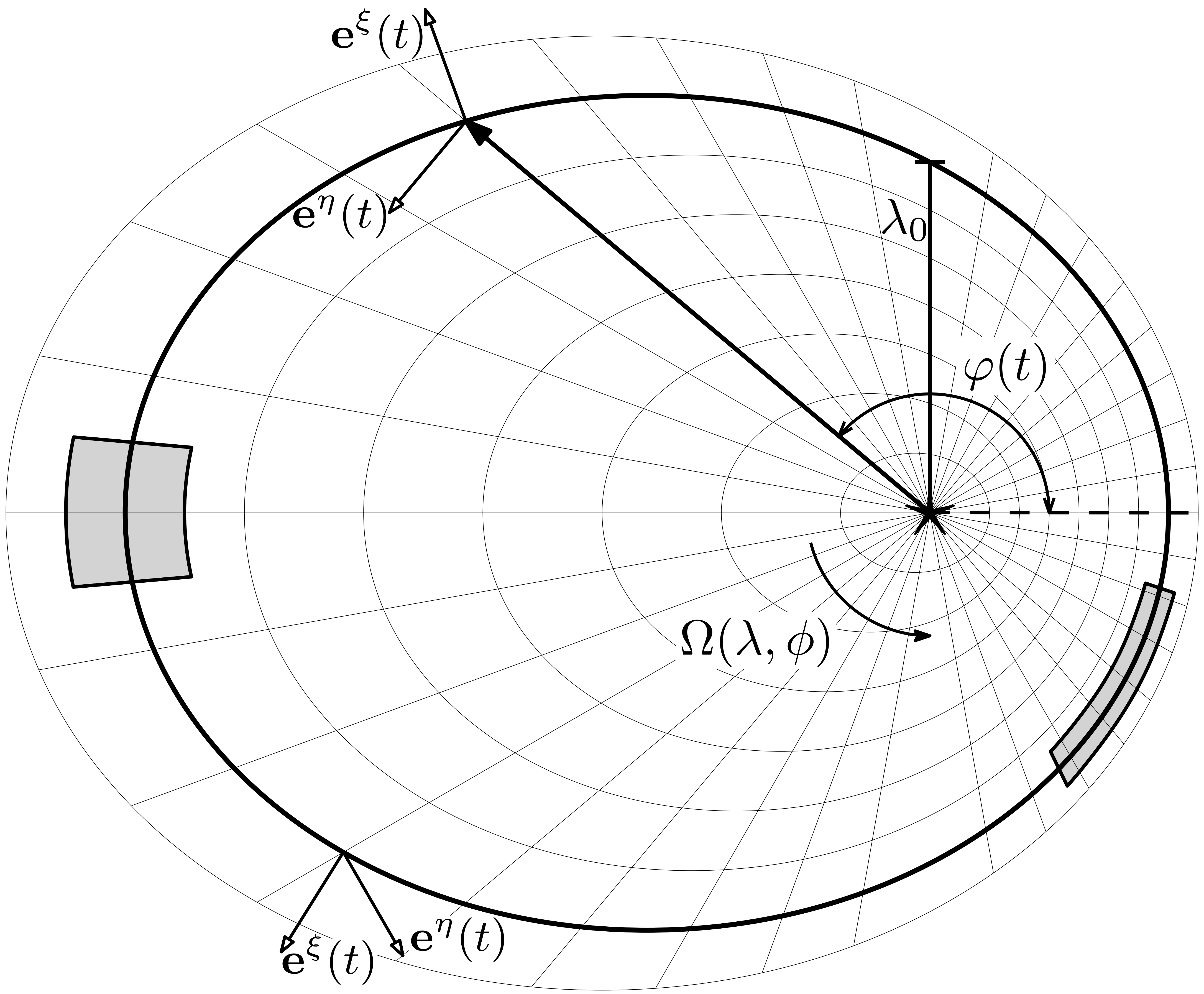
Local “Shearing Box” Model of a Circular Disc



- Useful for studying local disc phenomena
  - Computationally cheaper
  - More experimental control
- Enter a co-orbiting and rotating frame at R_0
- Make an expansion in the local coordinates

$$D\mathbf{v} = -\frac{1}{\rho}\nabla p - \underbrace{2\Omega_0\hat{\mathbf{z}} \times \mathbf{v}}_{\text{Coriolis}} + \underbrace{2q\Omega_0^2 x\hat{\mathbf{x}} - \Omega_0^2 z\hat{\mathbf{z}}}_{\text{Effective Tidal Acceleration}}$$

Local Model of an *Eccentric* Disc



- New generalised coordinates for the disc, (λ, ϕ, z)
  - Parameterised by $e(\lambda)$ & $\phi_0(\lambda)$
- Enter a co-orbiting frame $(\lambda_0, \phi(t), 0)$
- Expand in the local coordinates (ξ, η, ζ)
  - Time-varying metric in a non-orthogonal local coordinate system
- *Generalisation of the cartesian shearing box!*

Governing Equations for the Local Eccentric Disc Model

Numerical Solution:

- Simplify by assuming *local* axisymmetry: 2.5D
- Use *co-variant* azimuthal momentum equation
- 2nd Order FVM Solver

$$D\rho = -\rho(\Delta + \partial_\xi v^\xi + \partial_z v^z)$$

$$Dv^\xi = -\frac{1}{\rho} g^{\lambda\lambda}\partial_\xi p - 2\Gamma^\lambda_{\lambda\phi}\Omega v^\xi - 2\Gamma^\lambda_{\phi\phi}\Omega v^\eta$$

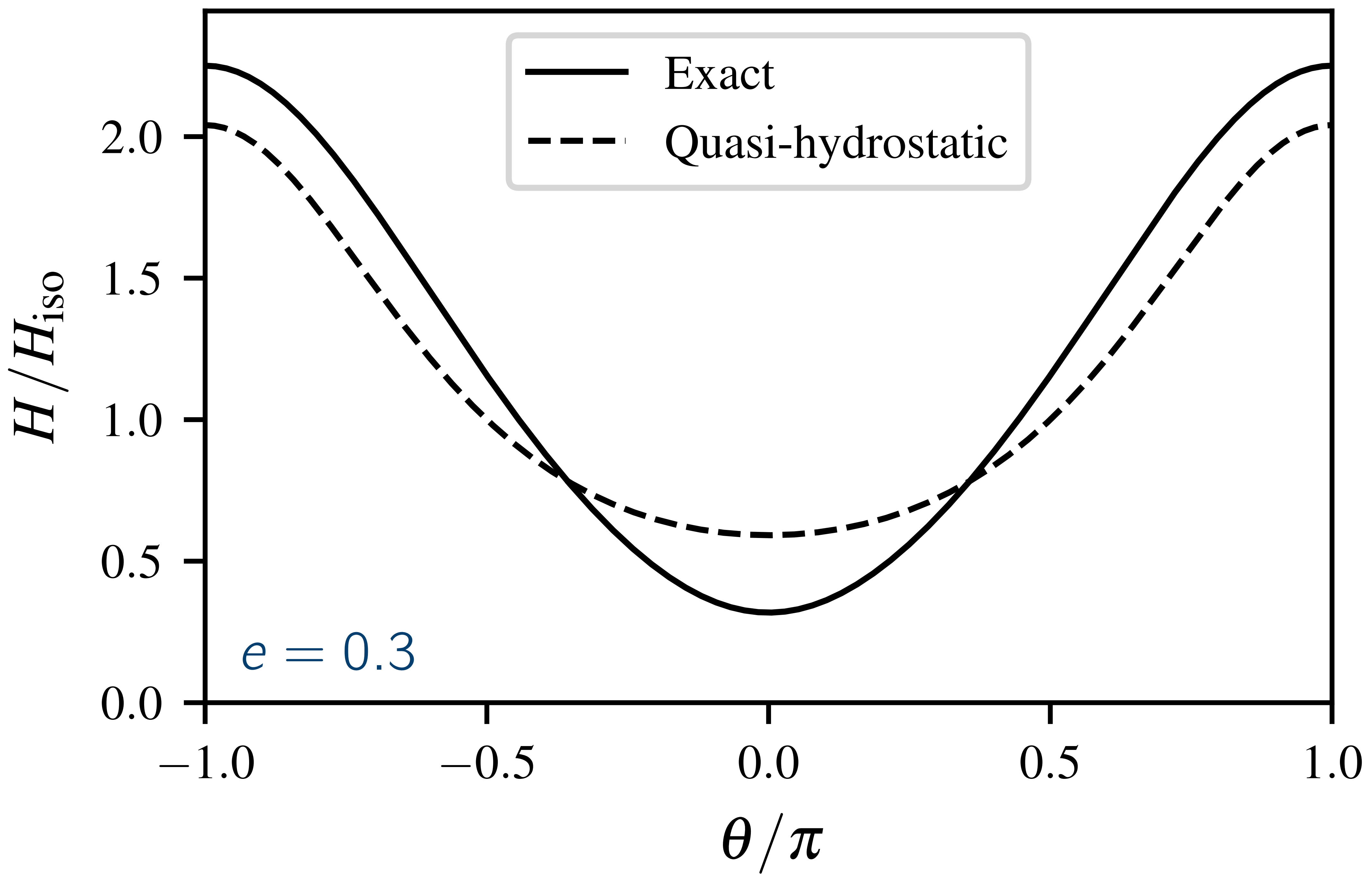
$$Dv_\eta = \left(g_{\phi\phi}\Omega_\lambda + \Omega\left(g_{\phi\phi}\Gamma^\phi_{\lambda\phi} - g_{\lambda\lambda}\Gamma^\lambda_{\phi\phi} - g_{\lambda\phi}\Gamma^\phi_{\phi\phi}\right)\right)v^\xi + g_{\phi\phi}\Omega_\phi v^\eta$$

$$Dv^z = -\Phi_2 z - \frac{1}{\rho}\partial_z p$$

$$Dp = -\gamma p(\Delta + \partial_\xi v^\xi + \partial_z v^z)$$

$$\Delta \equiv J^{-1}\partial_\phi(J\Omega)$$

Horizontally Invariant Laminar Solutions



- Globally-stationary solutions appear locally as a vertical "breathing" mode
- Similarity solution in $\zeta \equiv z/H$

$$\ddot{H} = -\Phi_2 H + \frac{c_s^2}{H}$$

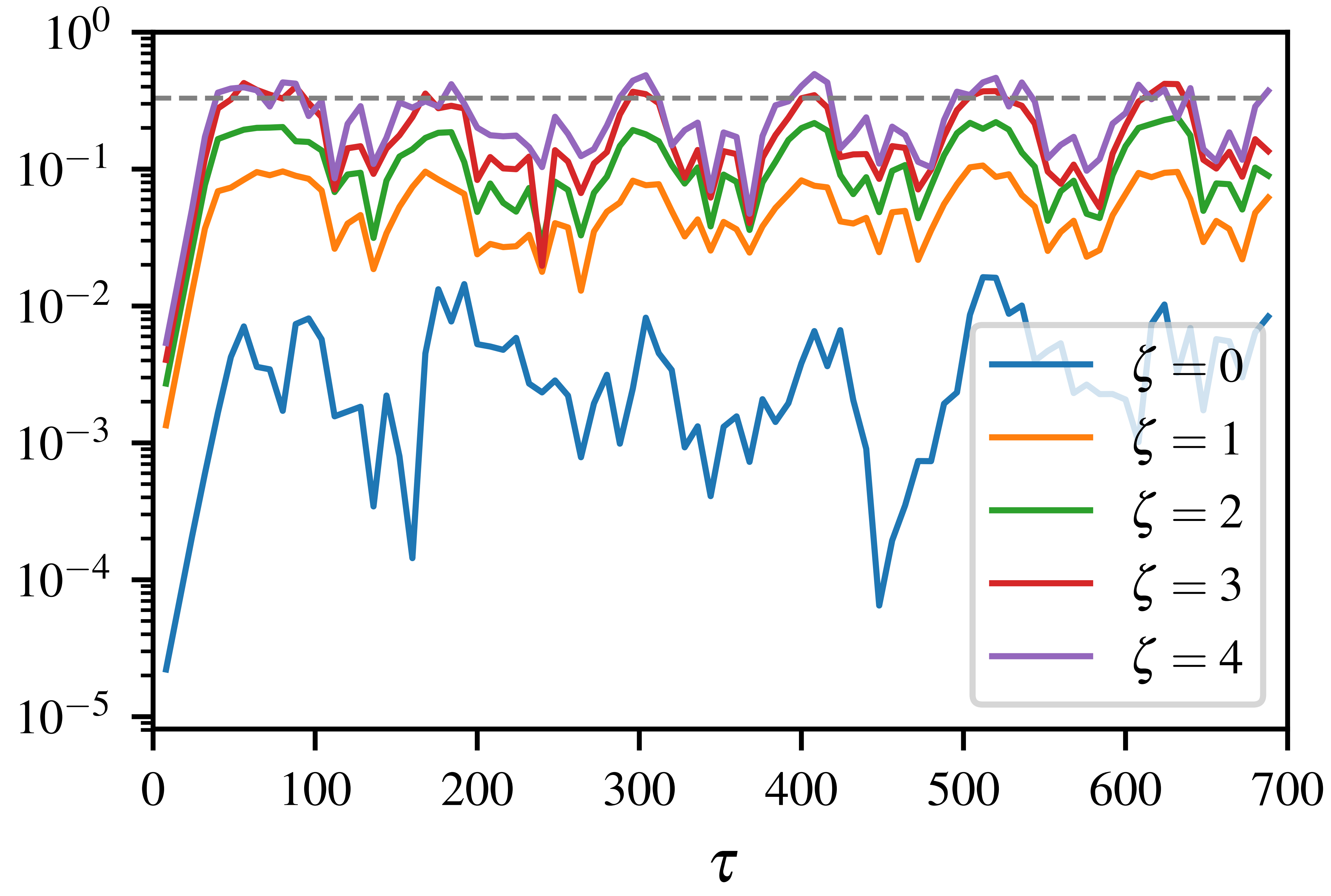
- Extreme convergence near pericentre with increasing eccentricity
- Transform equations into ζ coordinate

➡ Stationary equilibrium solution!

Parametric Instability of Inertial Waves

- Small-scale inertial waves couple with the eccentric mode
  - Resonance criterion: $\omega = \dfrac{m}{2}\Omega$
- Strongest resonance corresponds to a radial standing wave with $k_\xi = 3/2$
- Linear growth rate for an isothermal disc: $\sigma = 3/4$
  - vs Papaloizou (2005) *without* vertical structure, $\sigma = 3/16$
- Never previously been observed in numerical simulations
  - Global simulations were either 2D or had too limited of spatial resolution

Nonlinear Saturation by Inertial Wave Breaking



$e = 0.02$ ξ

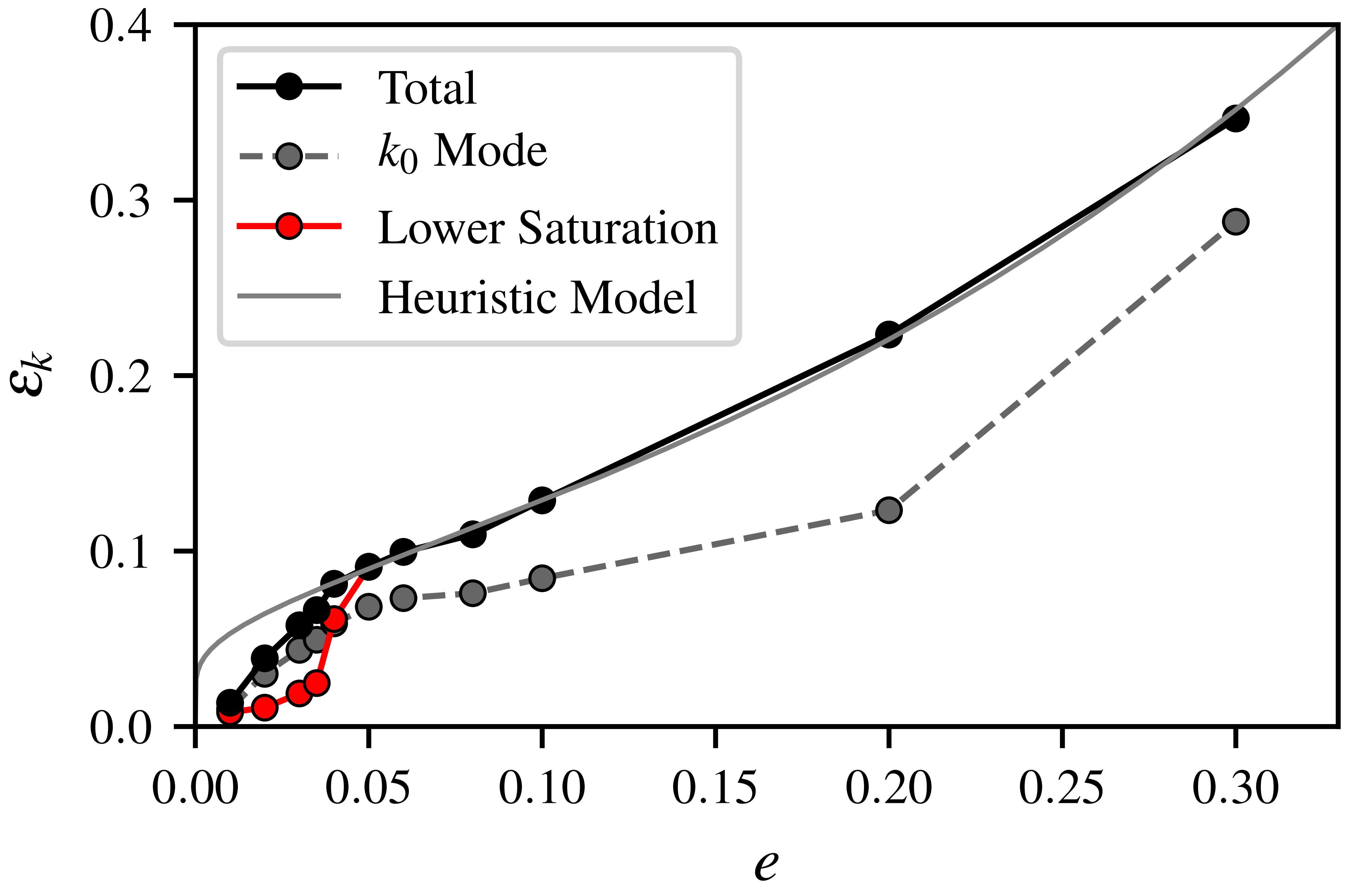
- Initialise the simulation with the exact linear inertial perturbation
- Observe linear growth in agreement with theory
- Inertial waves break when they locally violate the Rayleigh stability criterion:

$$\hat{v}^\xi_{\text{crit}} = \omega / k_\xi = 1/3$$

- From the profile of the vertical mode,

$$\zeta_{\text{crit}} = \frac{\omega}{k_\xi \hat{v}^\xi_0} e^{-\sigma t}$$

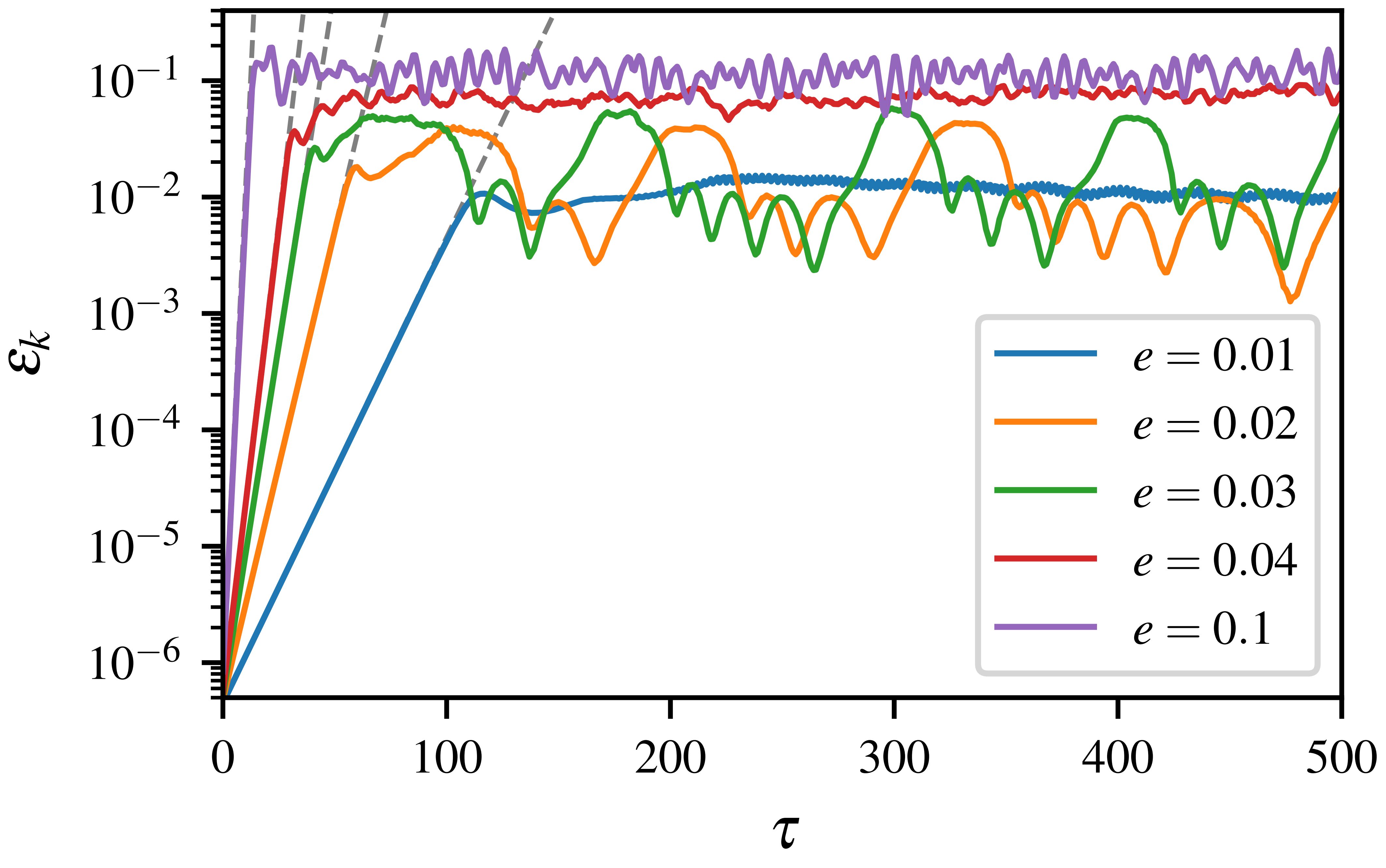
Modelling Energy Transfer at Saturation



Wienkers & Ogilvie, 2017

- Assume statistically time-stationary turbulence in spectral equilibrium
- Vertical energy balance:
  - Energy injection from eccentricity into k_0
  - Turbulent dissipation localised above some ζ_{crit}
    - Below ζ_{crit} inertial waves remain linear
    - Far above ζ_{crit} the density is too low
- Typical eccentricity decay times around 50–100 $\mathcal{P}$

Generation of Limit Cycles by Zonal Flows



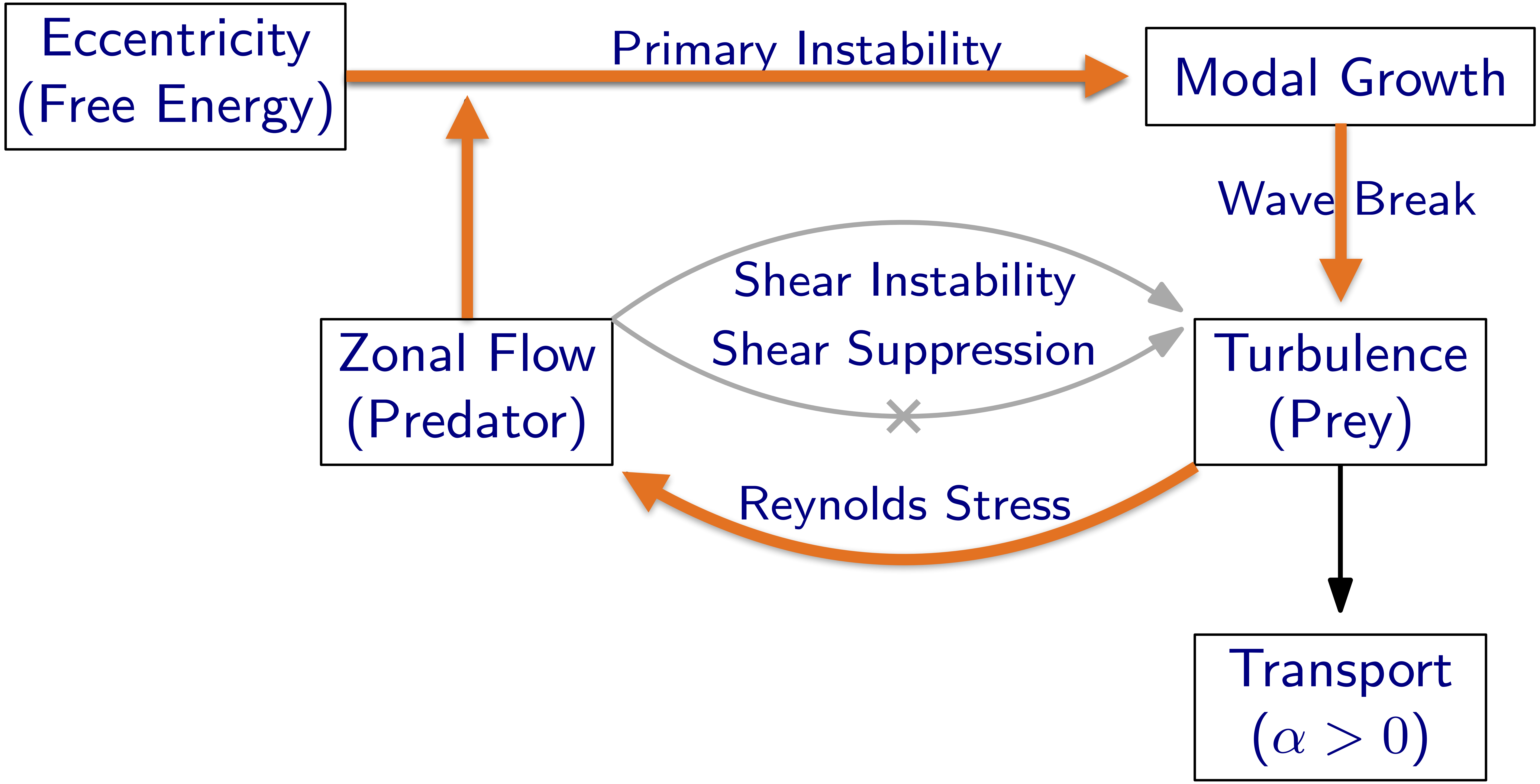
- Disagreement for $e \lessapprox 0.03$ attributed to a sub-critical Hopf bifurcation due to *zonal flows*
- Background zonal shear modifies epicyclic frequency

$$\kappa^2 = 2\Omega \left(\frac{1}{2}\Omega + R\partial_\xi v_{\text{zonal}} \right)$$

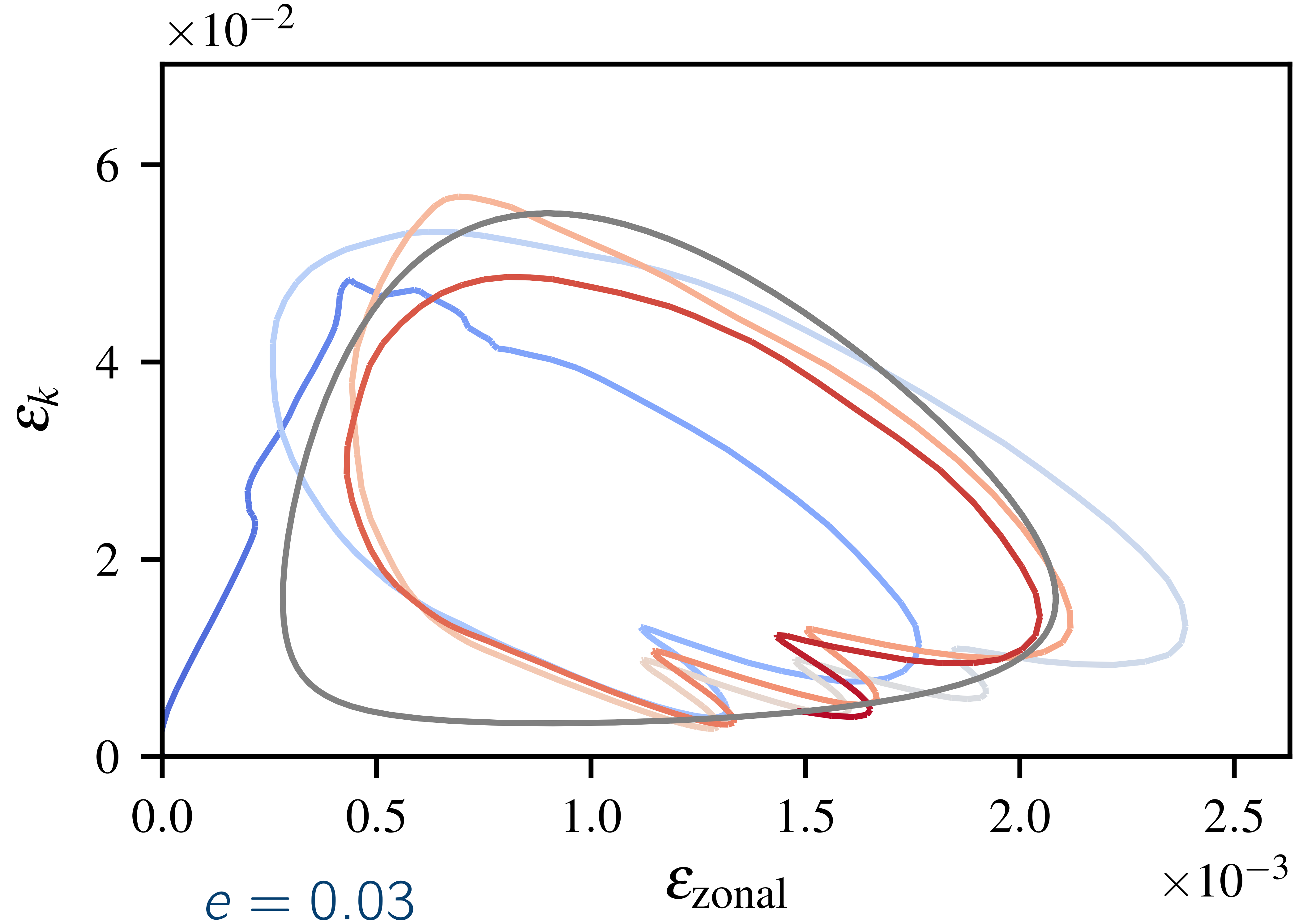
➡ Knocks off of parametric resonance!

- The zonal flows are generated by the saturated turbulent stresses, and so are self-regulating

Overview of the Energy Pathway & Feedback



Activator-Inhibitor Model of the Feedback



$e = 0.03$

- Energy growth based on Floquet analysis giving finite bandwidth of instability (Ogilvie & Barker, 2014)
- Parameterised zonal flow growth and large-scale damping:

$$\partial_\tau \varepsilon_k = \sigma_k \varepsilon_k - \frac{\sigma_k}{2w^2} \varepsilon_k v_{\text{zonal}}^2$$

$$\partial_\tau v_{\text{zonal}} = -c_{\text{damp}} v_{\text{zonal}} + c_{\text{eat}} \sqrt{\varepsilon_k}\, v_{\text{zonal}}$$

- Cannot model the short-circuit feedback from the shear instability of zonal flows directly feeding into turbulence

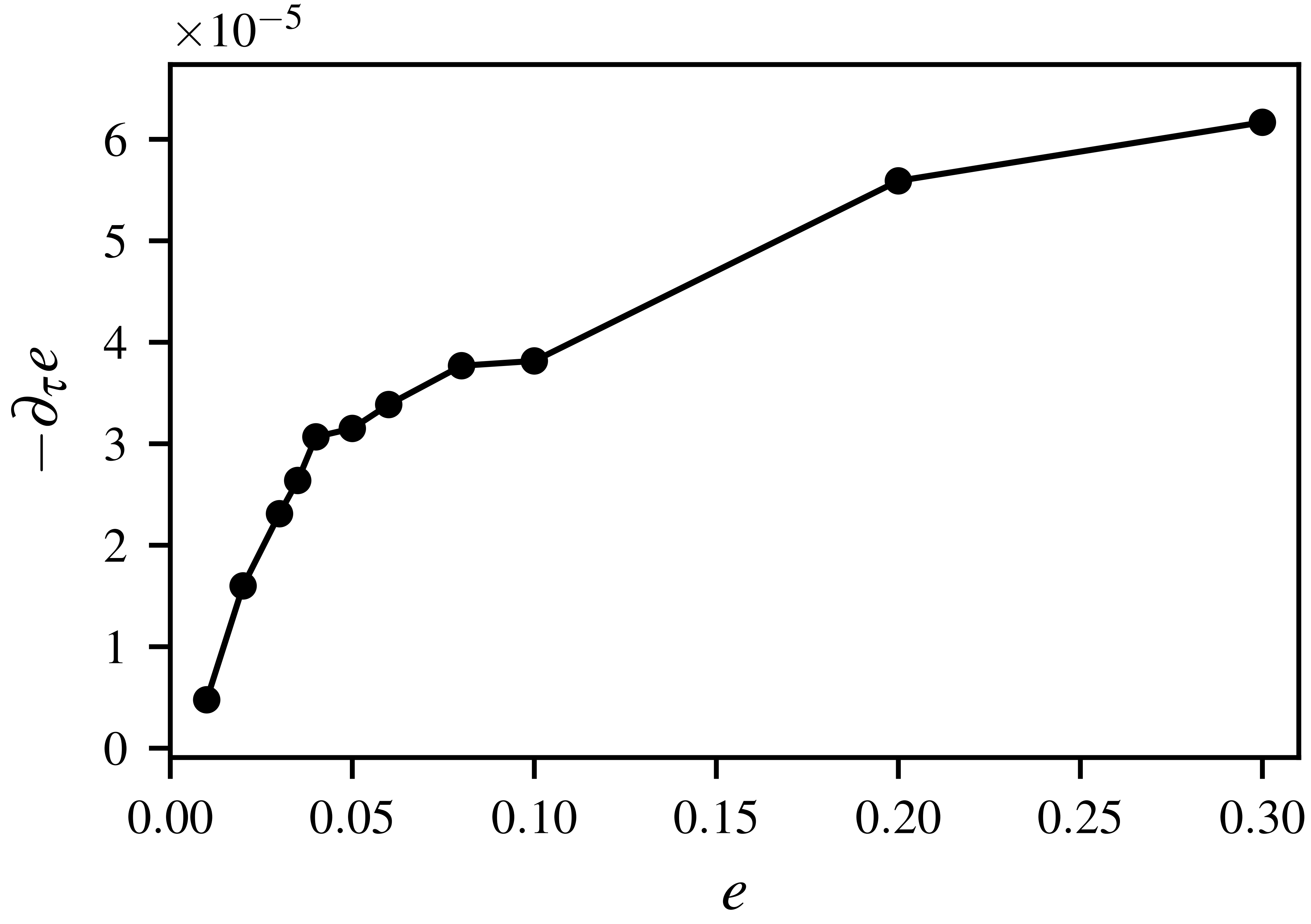
Conclusions

- Generalised the popular cartesian shearing box model to an eccentric disc
- Vertical disc structure is critical to both linear stability and saturation
  - Physically separates the energy sourcing near the mid-plane and localises wave-breaking around ζ_{crit}
- Rapid decay of eccentricity during saturation and self-regulation by zonal flows results in intermittency suggestive of outbursts
- This parametric instability may also be active in the MRI stable regions of protoplanetary and protostellar discs

Acknowledgements

Questions?

Eccentricity Decay Rate



- Assume a steady alpha-disc radial profile,

$$\Sigma \propto r^{-3/4}$$

$$P \propto r^{-3/2}$$

- The $T^{\lambda\phi}$ stress terms then cancel exactly giving

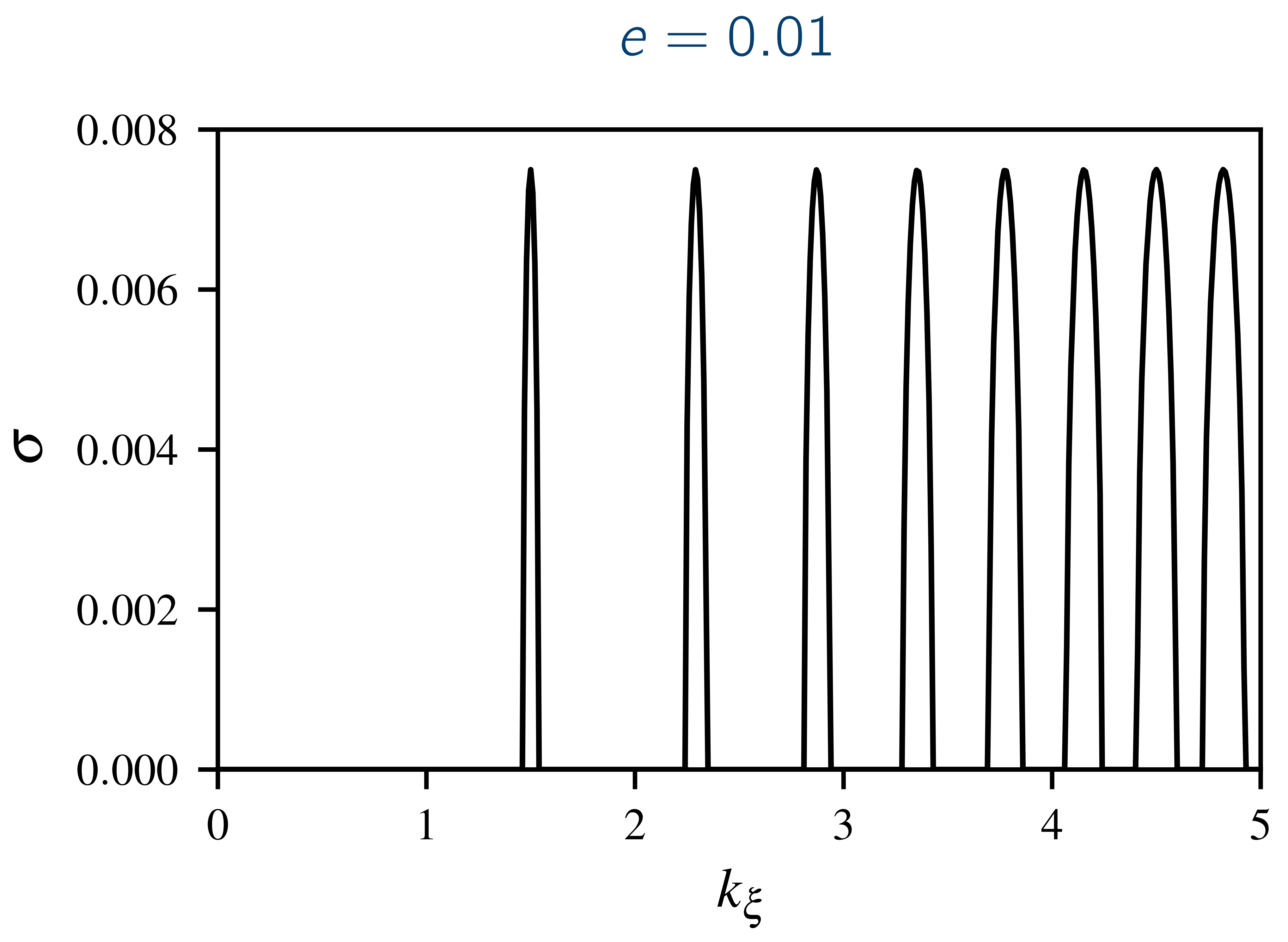
$$\ell_0 \mathcal{M} \partial_t e = \iint J \sin\theta \left(-\frac{1}{2} R_\lambda T^{\lambda\lambda} + R^2 T^{\phi\phi} \right) \Omega \, \mathrm{d}t \, \mathrm{d}z$$

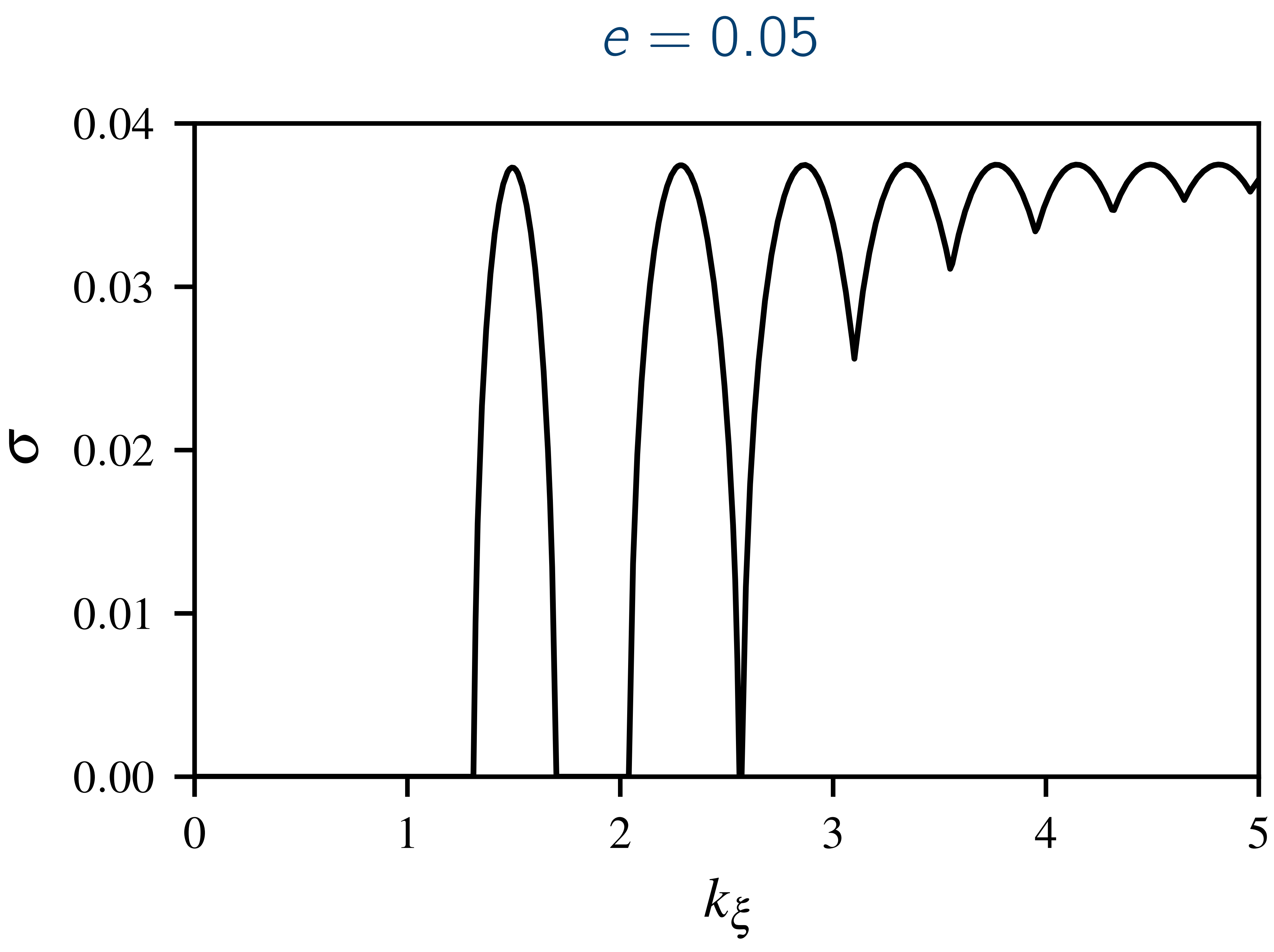
Alpha Viscosity

To generalise the alpha-viscosity to an eccentric disc, the transport of angular momentum must be described by the internal torque,

$$\alpha = -\frac{\iint J R^2 T^{\lambda\phi} \, \mathrm{d}\varphi \, \mathrm{d}z}{\iint J \lambda_0 p \, \mathrm{d}\varphi \, \mathrm{d}z}$$

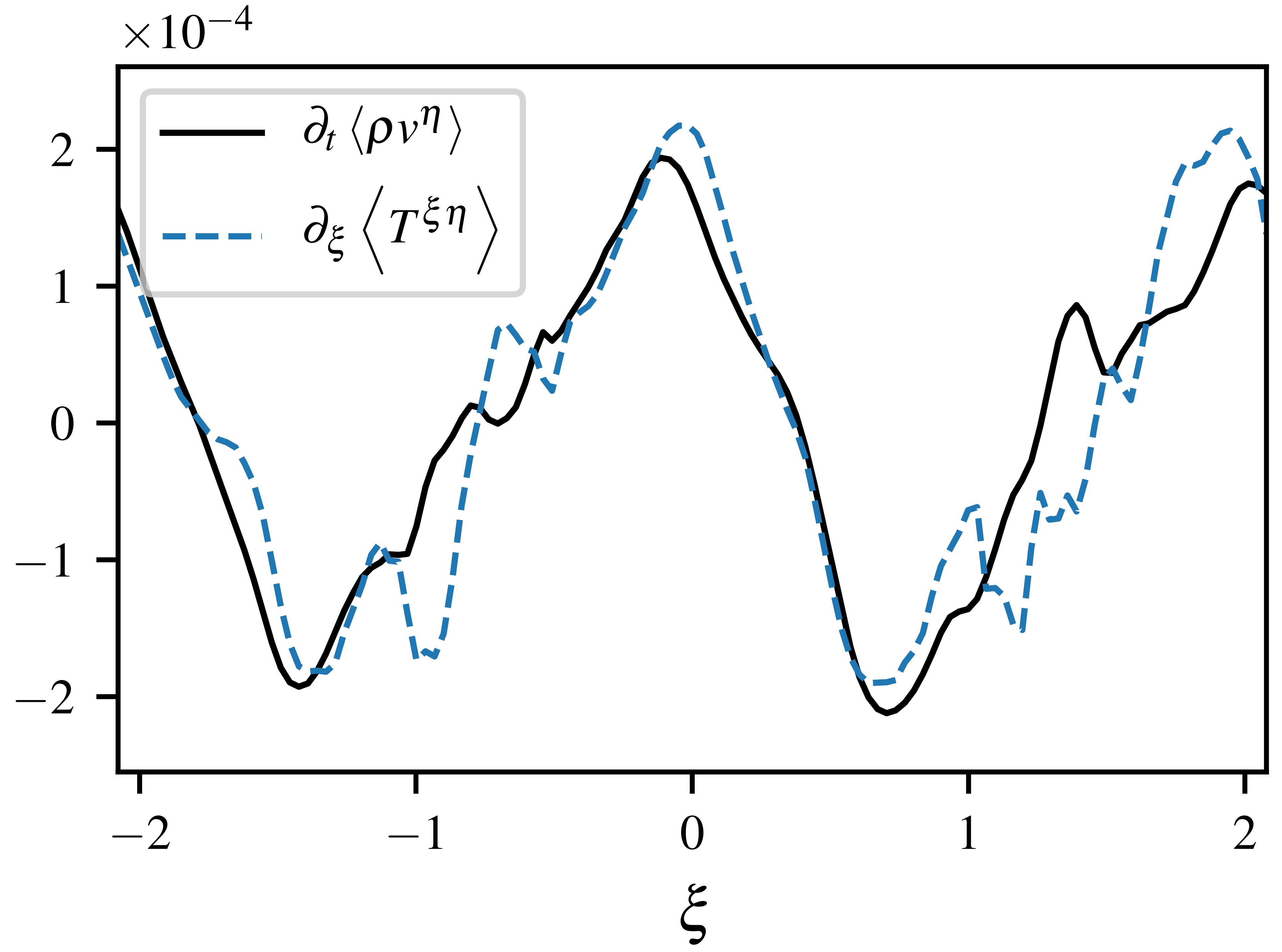
Floquet Stability Analysis

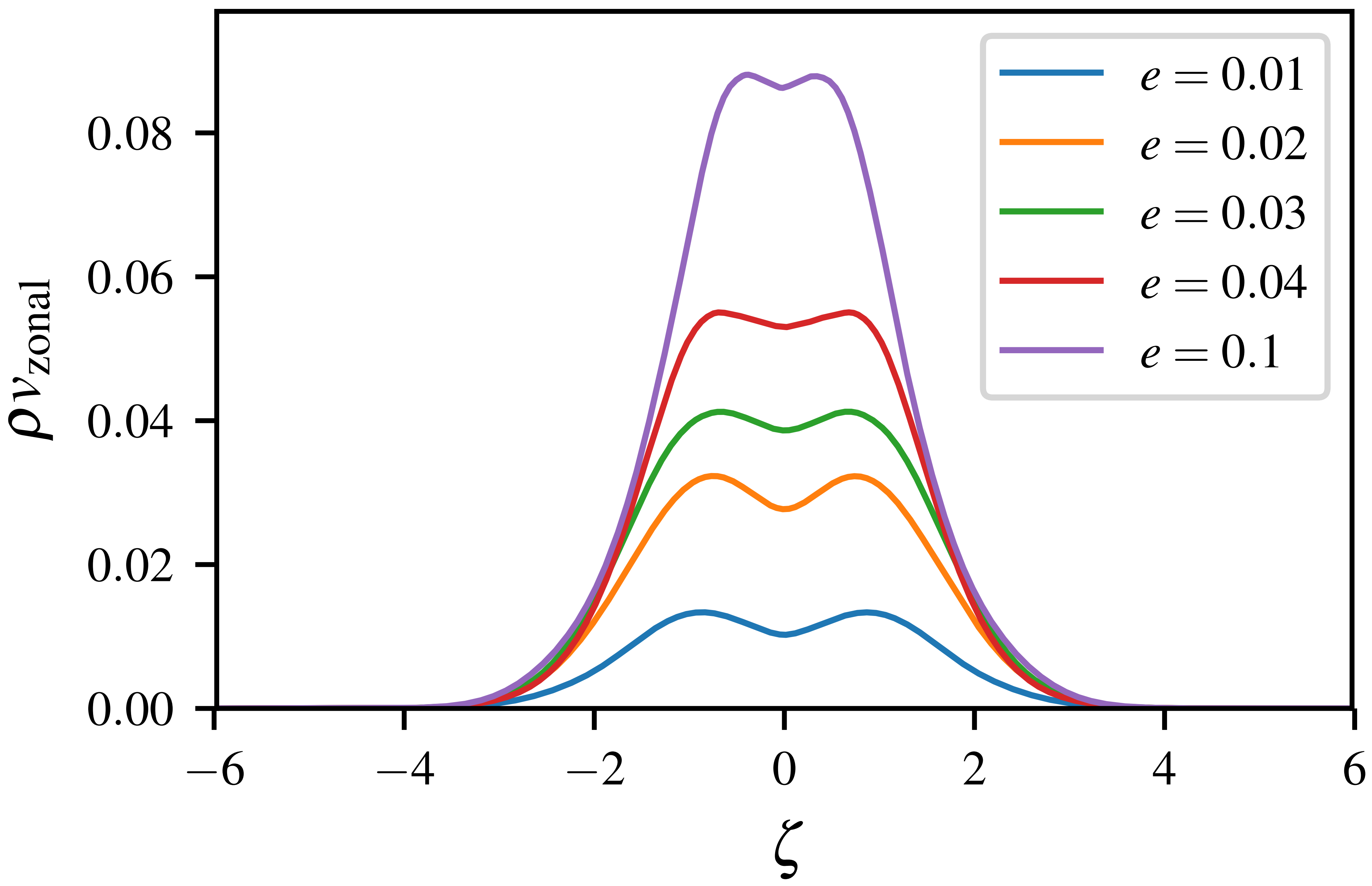




Barker & Ogilvie, 2014

Structure of Zonal Flows





Phase Slices of the Hopf Bifurcation