A Brief Review of Localisation Effects in Finite-size Parametric Subharmonic Instability

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Internal waves have recently been shown to be a crucial component of energy transport and mixing in the oceans. A number of mechanisms are known to facilitate the direct cascade of energy from the large scales of internal waves down to small-scale mixing. The parametric subharmonic instability (PSI) is perhaps the most studied mechanism recently, but is only well established for infinite plane waves in the inviscid limit as well as with limited viscosity (Koudella & Staquet 2006). Naturally occurring internal waves are of course finite width, and often of complex-valued envelope profile (Thomas & Stevenson 2006). It has further been shown experimentally (Bourget *et al.* 2013), and in limiting cases theoretically (Karimi & Akylas 2014), that finite beam width effects are significant for the ability of PSI to extract and transfer energy to smaller scales. Yet we show here that there is little agreement to the extent of this effect.

Thus we present the results of a series of numerical studies investigating the width effects on the threshold amplitude for PSI, and compare these to two independent analyses (Bourget *et al.* 2014; Karimi & Akylas 2014). Finally, we present an investigation of the effects of mean flow and circulation on PSI and explain the influence on the evolution of the subharmonic daughter frequencies using a time-frequency analysis.

Key words: geophysical and geological flows — stratified flows — internal waves: parametric instability

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The transfer of energy from large scales to smaller scales is critical for the dynamics of many geophysical and astrophysical processes. Understanding these mechanisms is thus essential to building accurate geophysical models. Stratified or rotating fluids can store this large-scale energy as internal waves which are known to be an effective means of momentum transport in the atmosphere (Alexander *et al.* 2006) as well as delivering critical perturbations in stars and protostellar accretion disks (Ogilvie 2005; Barranco & Marcus 2005). In the oceans, internal waves are believed to be a key component in the thermohaline "ocean energy budget" necessary for meridional overturning of the stratified waters via turbulent mixing (Munk & Wunsch 1998), as well as a general means of dissipating tidal energy (see Garrett & Kunze (2007) for a review).

While energy ray tracing methods are well established (Sutherland 2009), there is still much to learn about how energy is practically transported from large-scale internal waves to smaller scales and induces mixing, as well as where along the ray trajectory energy is deposited. Thus the threshold for instabilities capable of producing this direct cascade in

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the ocean is of increasing interest, allowing the location of topography or sources capable of supercritical forcing.

We will now focus attention on internal gravity waves in the oceans, which are borne out of a balance between inertia and buoyant forces from thermohaline stratification. These internal waves can be generated by tidal action over ocean ridges (Bell 1975) as well as flow across ocean topography (Laurent *et al.* 2003; Lamb 2004), which are radiated away, transporting energy and momentum with it. A number of nonlinear mechanisms are known to be able to dissipate the internal wave energy in the oceans. These include near-critical reflection (Dauxois & Young 1999), interaction with and scattering on small-scale topography (Kunze & Llewellyn Smith 2004; St Laurent & Garrett 2002), coupling with boundary layers and shear (Lamb 2014), and of course the subject of much attention recently: the Parametric Subharmonic Instability (PSI).

Internal gravity waves are known to be parametrically unstable to infinitesimal perturbations which grow to form temporal and spatial resonant triads with the producing primary wave beam Mied (1976); Klostermeyer (1982). This quadratic nonlinear instability of the Navier-Stokes equation, the Parametric Subharmonic Instability, produces two daughter waves with (in the inviscid limit) frequency equal to one half of the primary wave forcing frequency, as the name suggests. It is thus inherent to this instability to produce a direct cascade of energy from large scales ($\kappa_0 = |\mathbf{k}_0|$, where $\mathbf{k}_0 = (\ell_0, k_0)$ is the wave vector of the primary wave) to smaller scales (wavenumbers κ_1 and κ_2 of the subharmonic daughter waves). These resonant triad daughter waves satisfy the spatial resonance condition,

$$\mathbf{k}_0 = \mathbf{k}_1 + \mathbf{k}_2, \tag{0.1}$$

as well as temporal resonance,

$$\omega_0 = \omega_1 + \omega_2, \tag{0.2}$$

through which PSI shares energy between the three waves. It may be noted in Figure 1 that while the wavevectors form a triad, the group velocities, which dictate wave propagation perpendicular to the wavevectors across the horizontal, in general will not form a triad. Additionally, for these internal gravity waves, the dispersion relation,

$$|\omega|/N = |\sin\theta|, \tag{0.3}$$

relates the wave frequency, ω, with the direction of the wave vector, dictating the phase direction. Here θ is the energy propagation direction to horizontal which depends on the fluid stratification in terms of the buoyancy frequency, $N \equiv \sqrt{-g(\mathrm{d}\bar{\rho}/\mathrm{d}z)/\rho_0}$, wave frequency, and which is perpendicular to the wave vector.

Energy transfer rates due to PSI between an infinite primary plane wave beam and the daughter waves are well established (Koudella & Staquet 2006); however, recent experiments by Bourget *et al.* (2013) as well as theories using an energy approach by Bourget *et al.* (2014) and asymptotic stability analysis by Karimi & Akylas (2014) show a more complex picture for finite-width internal gravity wave beams. Further, the oceans are known to form finite-width beams through tidal interaction with ridges and steep slopes (Gostiaux & Dauxois 2007), and so much attention has turned to the effects of the finite size and envelope shape for the onset of PSI, to try and better understand the possible role in oceanic mixing.

The finite width of a wave beam increases its stability due to the advection of the daughter waves out of the triadic interaction zone of the primary wave beam before they can extract substantial energy. This interaction time scales directly with the perpendic-

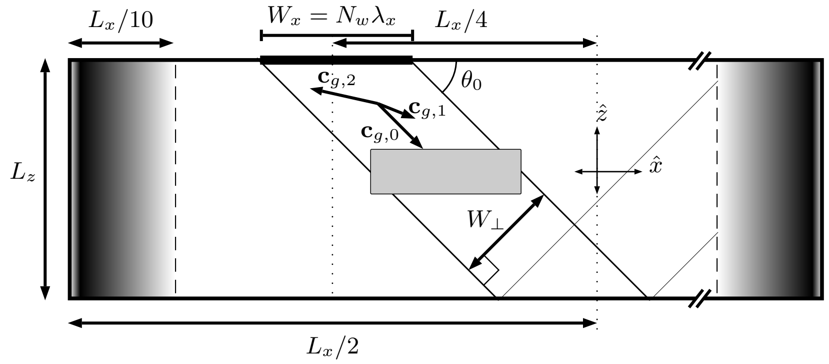

FIGURE 1. Diagram of the general domain setup for numerical experiments.

ular beam width, $W_\perp$ as

$$\tau_{\text{adv},i} = \frac{W_\perp}{c_{g,i}\sin(\theta_0 - \theta_i)}, \tag{0.4}$$

where $c_{g,i} = \sqrt{N^2 - \omega_i^2}/\kappa_i$ is the group velocity. In the inviscid limit, the daughters each have frequency $\omega_{1,2} = \omega_0/2$, and so the sine term is approximately $\omega_0/(2N)$ for the small ω/N used in numerics and experiments. The advection time across the inviscid beam for each daughter is then $\tau_{\text{adv}} \sim 2W_x/c_g$ for the horizontal component of width, $W_x = N_w\lambda_x$, of a beam carrying N_w wavelengths. As will be shown later, the instability is sensitive to the selection of the daughter frequencies with non-zero viscosity due to this term. In the end, a consequence of the shorter triad interaction is that less energy is able to be extracted from the primary wave beam and dissipated into smaller scales. The finite width of the wave beam is therefore a stabilising effect.

We perform these numerical studies complementary to the experimental setup and internal wave generator design of Mercier *et al.* (2010) in the horizontal configuration as described in Bourget *et al.* (2013). The location of the generator at the top of the domain and its phase directionality dictates a single beam propagation direction as opposed to the four directions suggested by (0.3) (Lighthill 2002). One shortcoming of this wave generator is that it is limited to real-valued beam envelopes. Sutherland (2013) posits that these sinusoidal wavetrains modulated by a localised envelope may not be entirely pertinent to oceanic waves. Thus we will also set up our analysis to consider general localised (complex-valued) beam profiles, motivated to study the Thomas & Stevenson (2006) and Jouve & Ogilvie (2014) similarity solutions for waves generated by barotropic tides acting over ocean topography and inertial waves, respectively. The breakdown of these more realistic geophysically-generated internal wave beams to smaller scales is critical in understanding oceanic mixing and the often mentioned ocean energy budget.

This paper seeks to rectify disagreements over the critical threshold amplitude for PSI, between the aforementioned theories, experiments, and simulations. We first construct and detail our numerical experiments and the spectral numerical method. We then present each method on equal footing, and compare the independent theories and experiments with numerical experiments. We finally draw some insight into these methods and propose possible corrections to account for shape and localisation effects in finite-size PSI.

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