The Role of Interactions between Waves and Baroclinic Critical Layers in Zombie Vortex Self-Replication

Chung-Hsiang Jiang[1], Suyang Pei[1],
Pedram Hassanzadeh[2],
Aaron Wienkers[1], Caleb Levy[1]
Philip S. Marcus[1]

[1]University of California, Berkeley
[2]Harvard University

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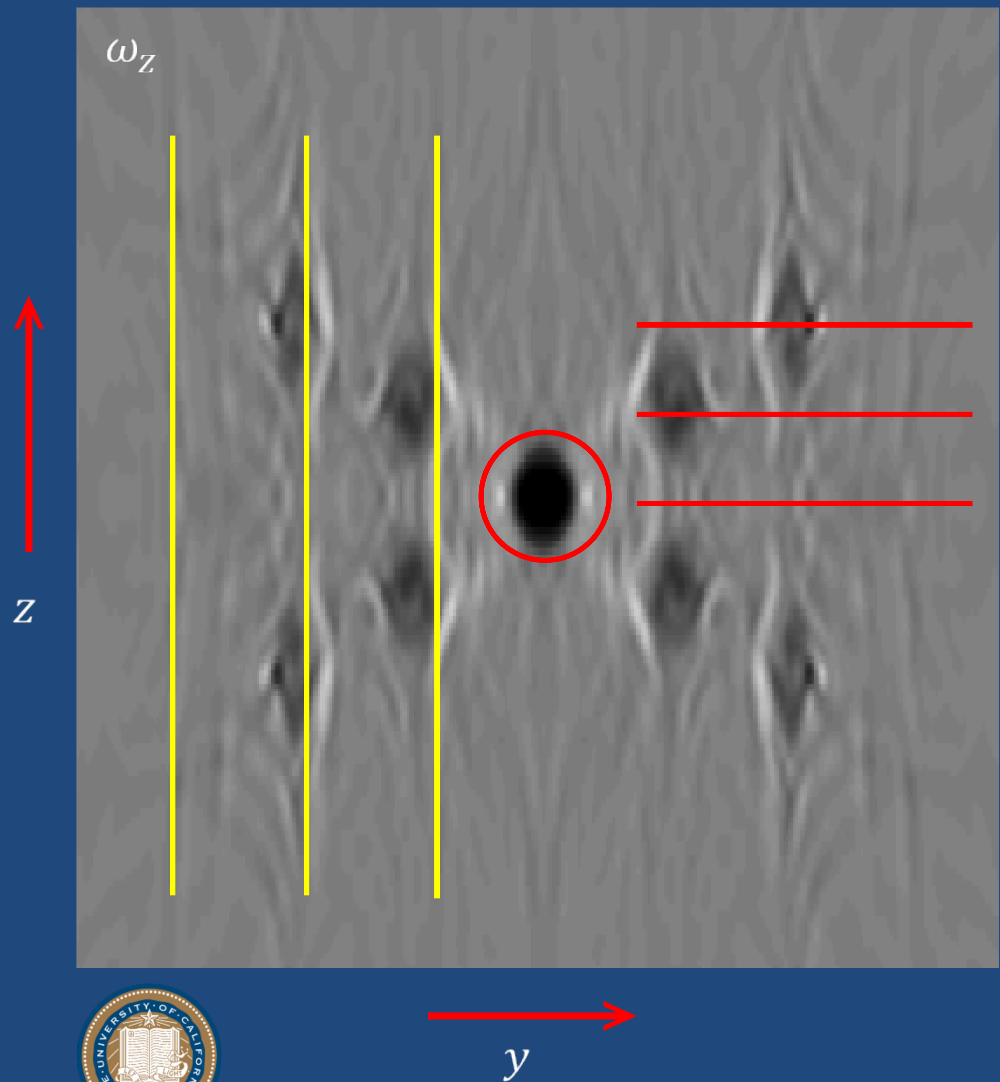
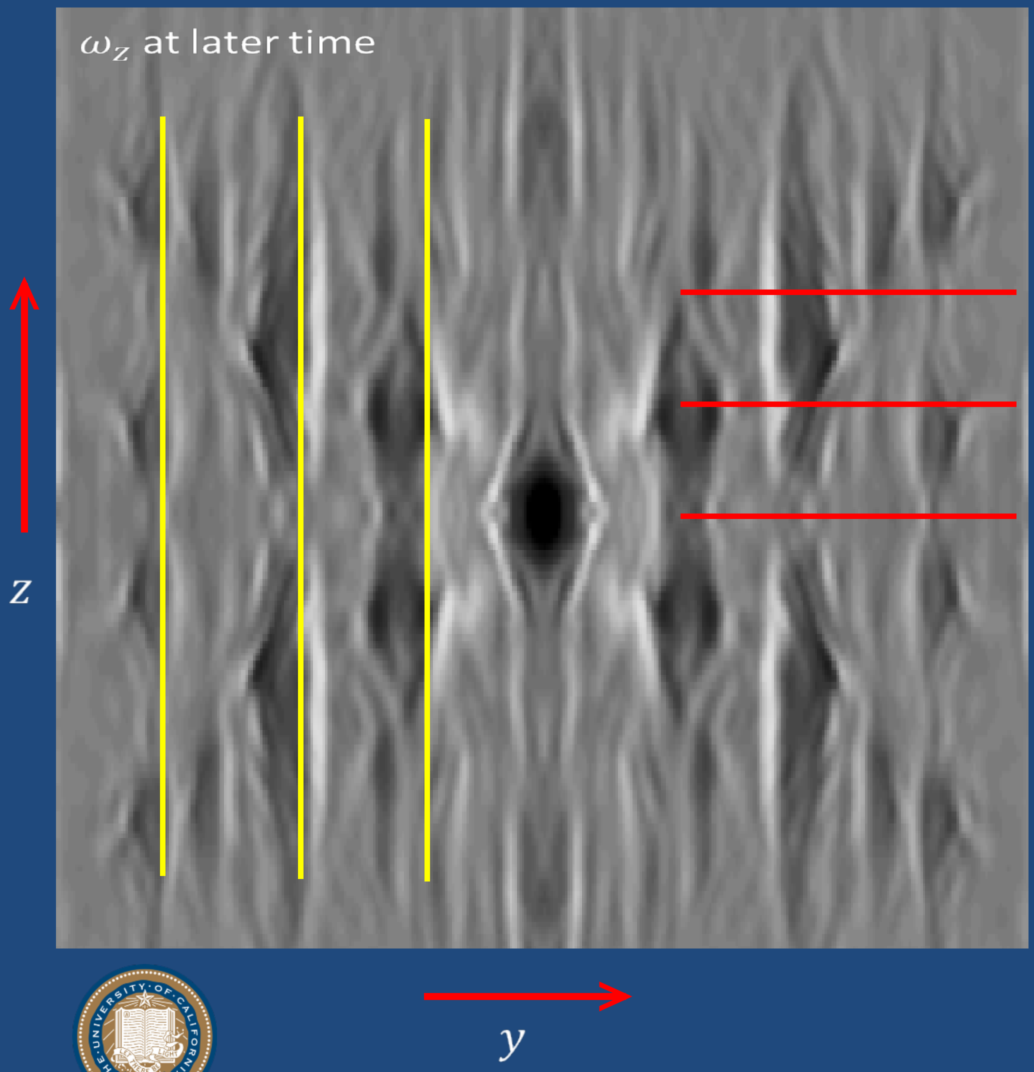
Motivation: Zombie Vortices in Plane Couette Flow Plus Stratification



- 3D Boussinesq equation
- Plane Couette flow with velocity $\overline{V_x} = \sigma\, y$
- Rotation $\Omega\, \hat{z}$
- Coriolis parameter $f = 2\Omega$
- Stably Stratified in vertical direction with Brunt-Väisälä frequency $N(z)$

Motivation: Zombie Vortices in Plane Couette Flow Plus Stratification



- Vortex lattice structure
- Three time scales characterized by $f \sim N \sim |\sigma|$
- Lack of explicit length scale
- But clearly see the uniform horizontal and vertical spacing

Goal

- Horizontal and vertical spacing of the lattice structures determined by locations where waves and critical layers have the same stream-wise wave number (n_x) interact at
- Need to derive dispersion relation of linear wave with large shear
- Large, we mean $f \sim N \sim |\sigma|$

Poincarè Waves with Variable $N(z)$ and No Shear

- Plane wave: $e^{i\,(\underline{k}\cdot\underline{x}-\omega t)}$, WKB in z-direction
- Dispersion Relation

$$\omega^2 = \frac{N^2(z)\,k_\perp^2}{k^2} + \frac{f^2\,k_z^2}{k^2}$$

- Local geometric form:

$$\tan^2\theta \approx (\omega^2 - f^2)/(N^2(z) - \omega^2)$$

- θ: the angle measured from horizontal plane
- Exact solution if $N(z) = N_0$ is constant

Poincarè Waves with Variable $N(z)$ and No Shear

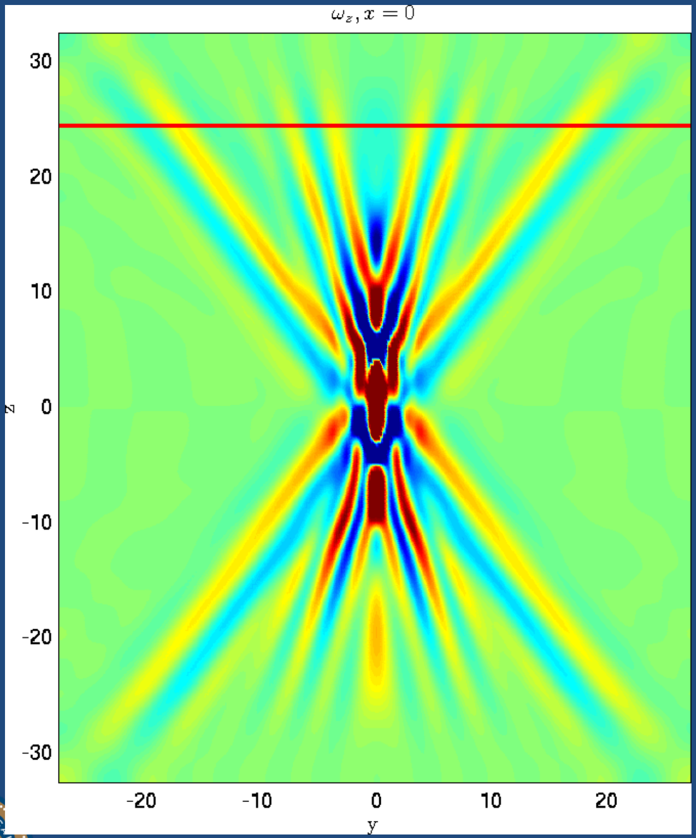
- Not depends on wave vector magnitude k but on wave vector direction only
- Allowed frequency ranges:
  - $f^2 \leq \omega^2 \leq N^2(z)$ internal gravity wave branch
  - $N^2(z) \leq \omega^2 \leq f^2$ inertial wave branch
- St. Andrew's cross in two dimensional and conical wave in three dimensional for constant N_0

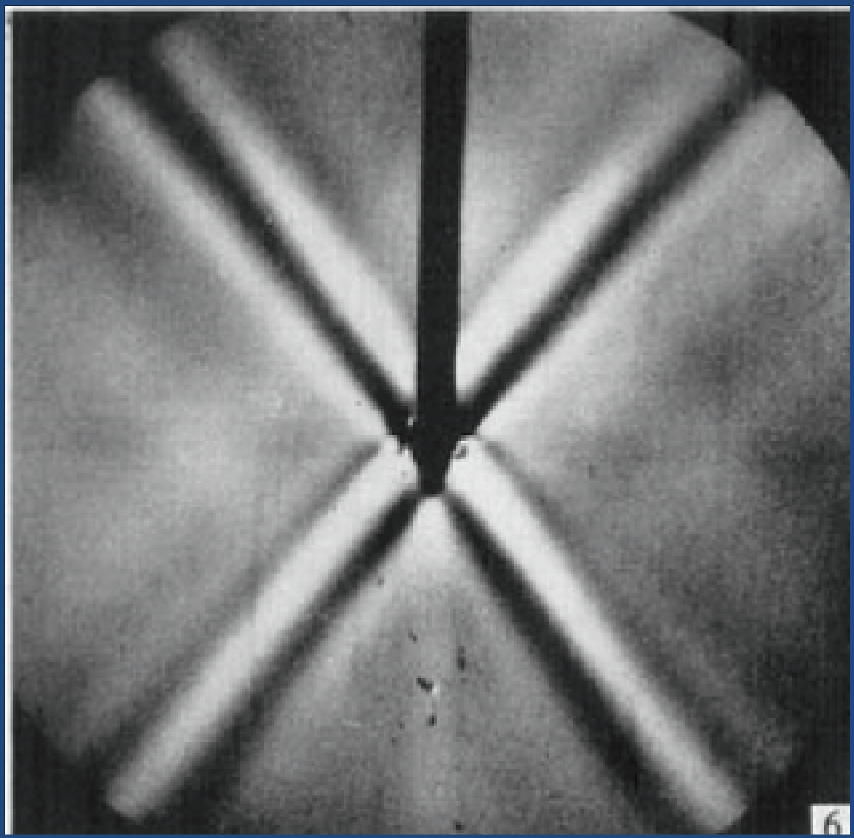
Poincarè Waves and No Shear

3D Nonlinear Simulation with a numerical wave generator
$N_0/f = 1/4$ **and** $\sigma = 0$

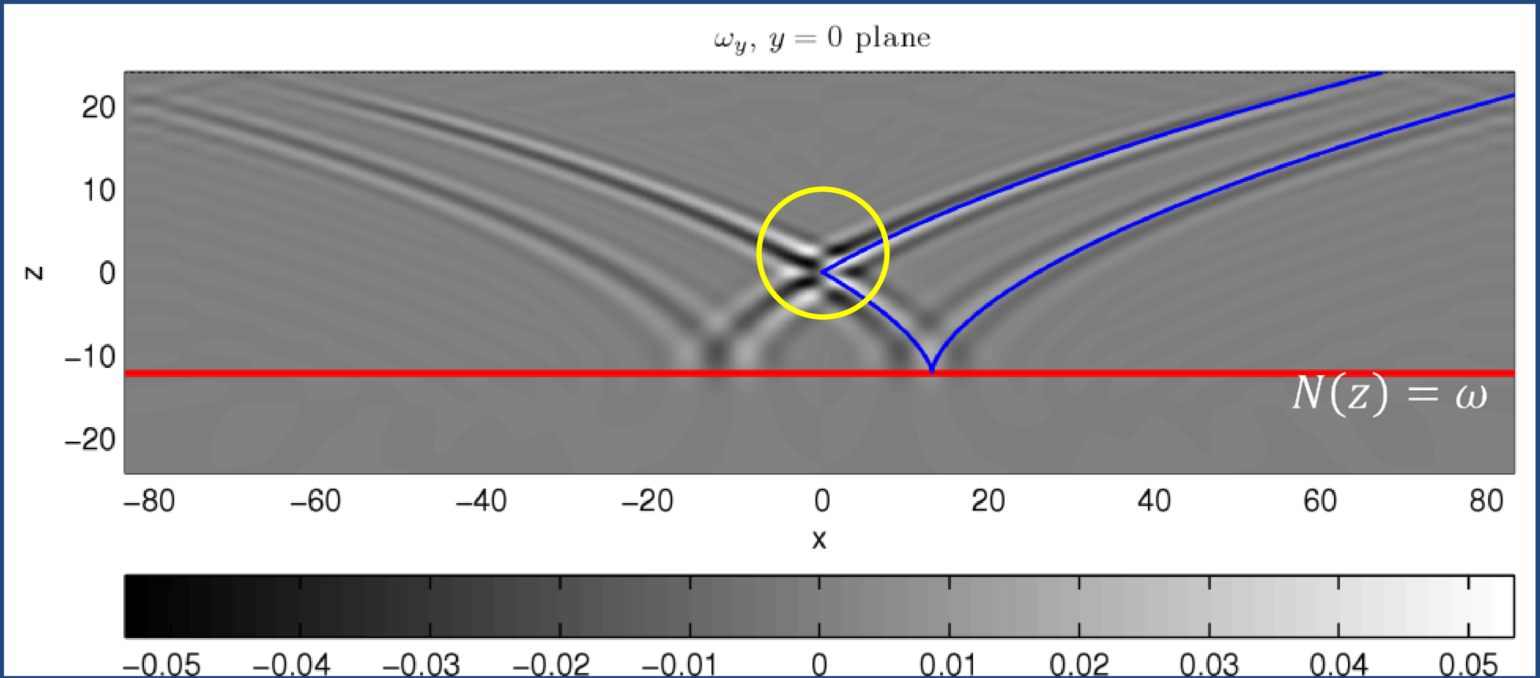
2D Experiment by Mowbray and Rarity
Constant N_0 **and** $f = 0, \sigma = 0$

z





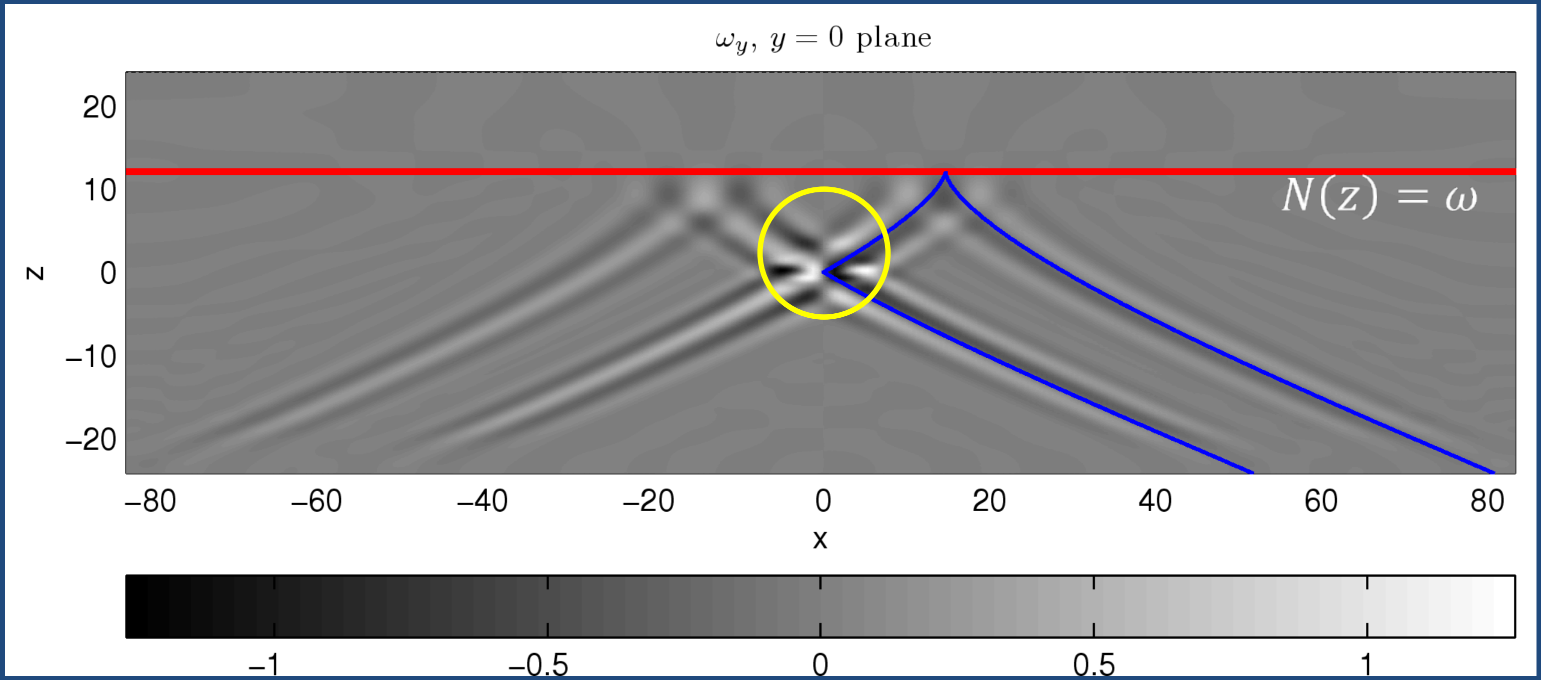
Poincarè Waves: No Shear with N(z)
Internal Gravity Wave Branch



- Wave generator forcing at constant $\omega = \omega_c$
- $f^2 \leq \omega^2 \leq N^2(z)$
- $N(z) = 2\ \omega_c\ (\ 1 + z/L_z)$
- $f = 0$

Blue: theoretical ray paths predicted by WKB theory
Red: Below it, $\omega > N(z)$, the forbidden region (evanescent waves)

Poincarè Waves: No Shear with N(z)
Inertial Wave Branch



- Wave generator forcing at constant $\omega = 3\ \omega_c$
- $N^2(z) \leq \omega^2 \leq f^2$
- $f = 2\ \omega_c \sqrt{8/3}$

Blue: theoretical ray paths predicted by WKB theory
Red: Above it, $\omega < N(z)$, the forbidden region (evanescent waves)

With Background Shear
WKB in both y- and z-directions

- Dispersion Relation: $\omega = \omega_0 + \underline{k} \cdot \underline{\overline{V}}$
- Intrinsic (un-perturbed) frequency

$$\omega_0^2 = \frac{N^2(z)\, k_\perp^2}{k^2} + \frac{f^2\, k_z^2}{k^2}$$

- Allowed(?) frequency ranges:
  - $f^2 \leq \omega_0^2 \leq N^2(z)$ internal gravity wave like
  - $N^2(z) \leq \omega_0^2 \leq f^2$ inertial wave like
- We always use $\underline{\overline{V}} = \overline{V}_x(y)\, \hat{x} = \sigma\, y\, \hat{x}$

With Background Shear
WKB in both y- and z-directions

- One thing that bothers us
- Exact solution for $k_x = 0$ and constant N_0 is known but NOT satisfied

$$\text{Exact form: } \omega_0^2 = \frac{N_0^2\, k_y^2}{k_y^2+k_z^2} + \frac{f(f-\sigma)\, k_z^2}{k_y^2+k_z^2}$$

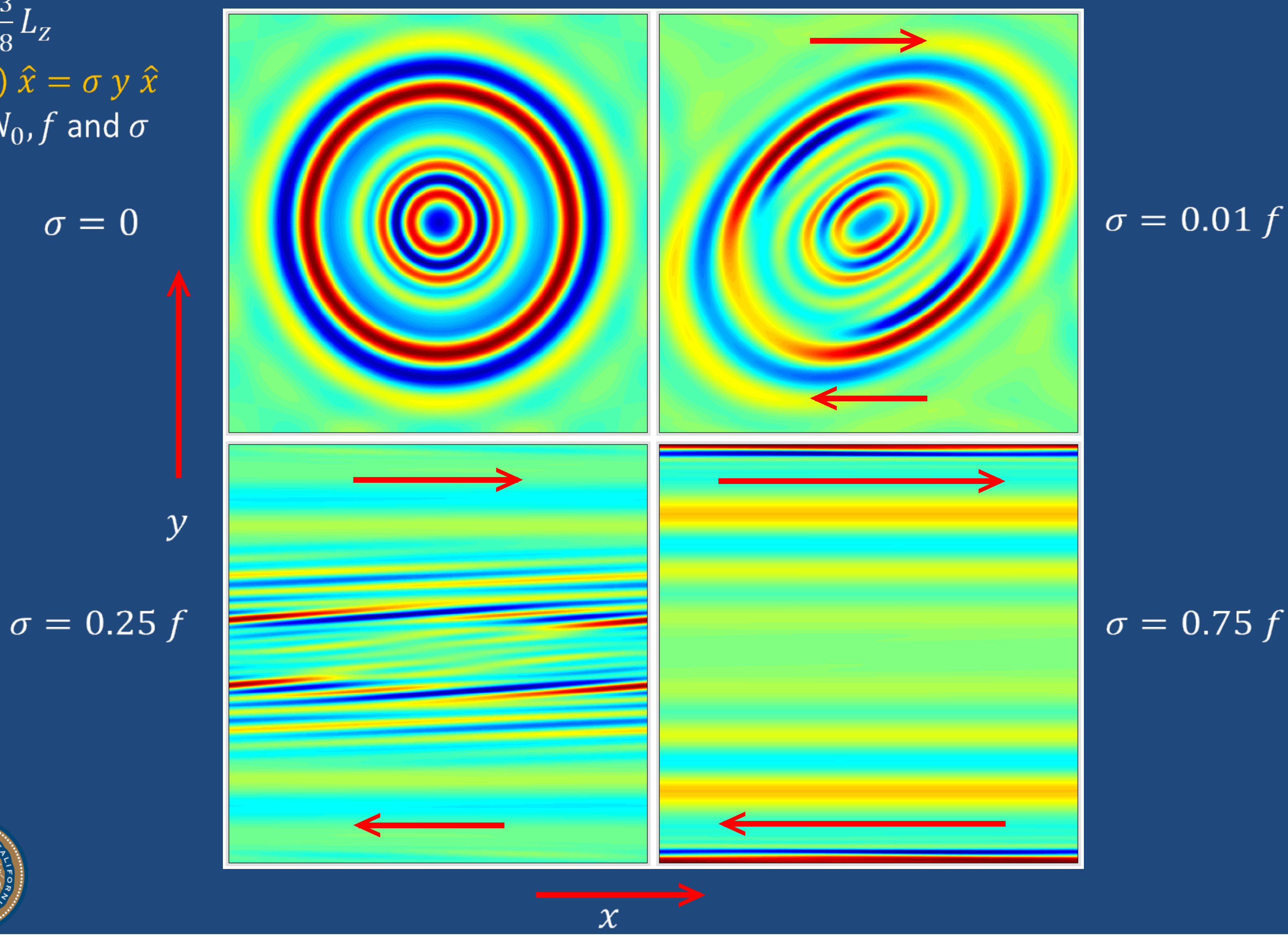
- Allowed frequency ranges:
  - $f(f-\sigma) \leq \omega_0^2 \leq N_0^2$
  - $N_0^2 \leq \omega_0^2 \leq f(f-\sigma)$
- More importantly, one more complexity , Critical Layer, might appear

Poincarè Waves (Inertial Wave Branch)
Shear Effect in Physical Space

ω_z at $z = \frac{3}{8} L_z$

$\underline{\overline{V}} = \overline{V}_x(y)\,\hat{x} = \sigma\, y\, \hat{x}$

Constant N_0, f and σ

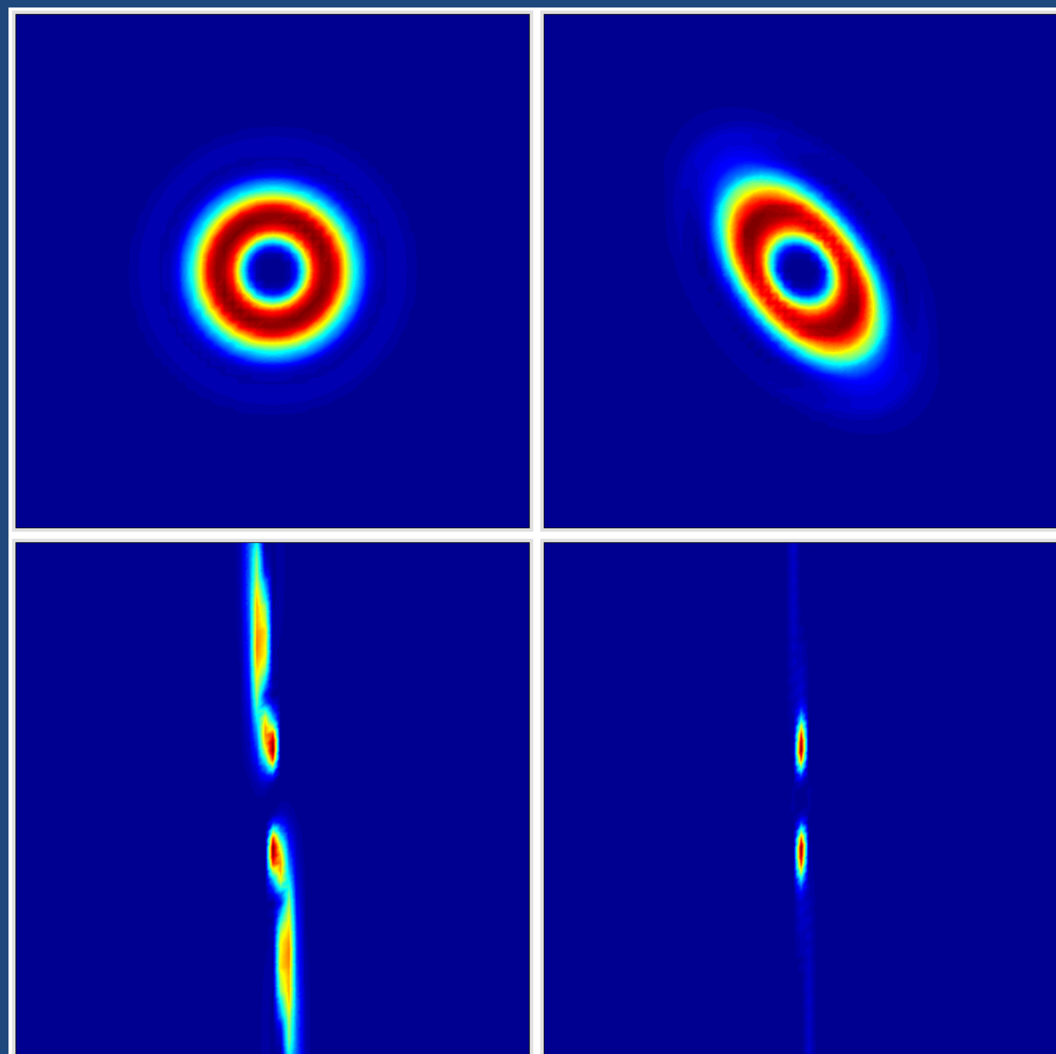


Poincarè Waves (Inertial Wave Branch)
Shear Effect in Fourier Space

$\widehat{\omega_z}$ at $z = \frac{3}{8} L_z$

$\sigma = 0$

$\sigma = 0.01\, f$



k_y

k_x

$\sigma = 0.25\, f$

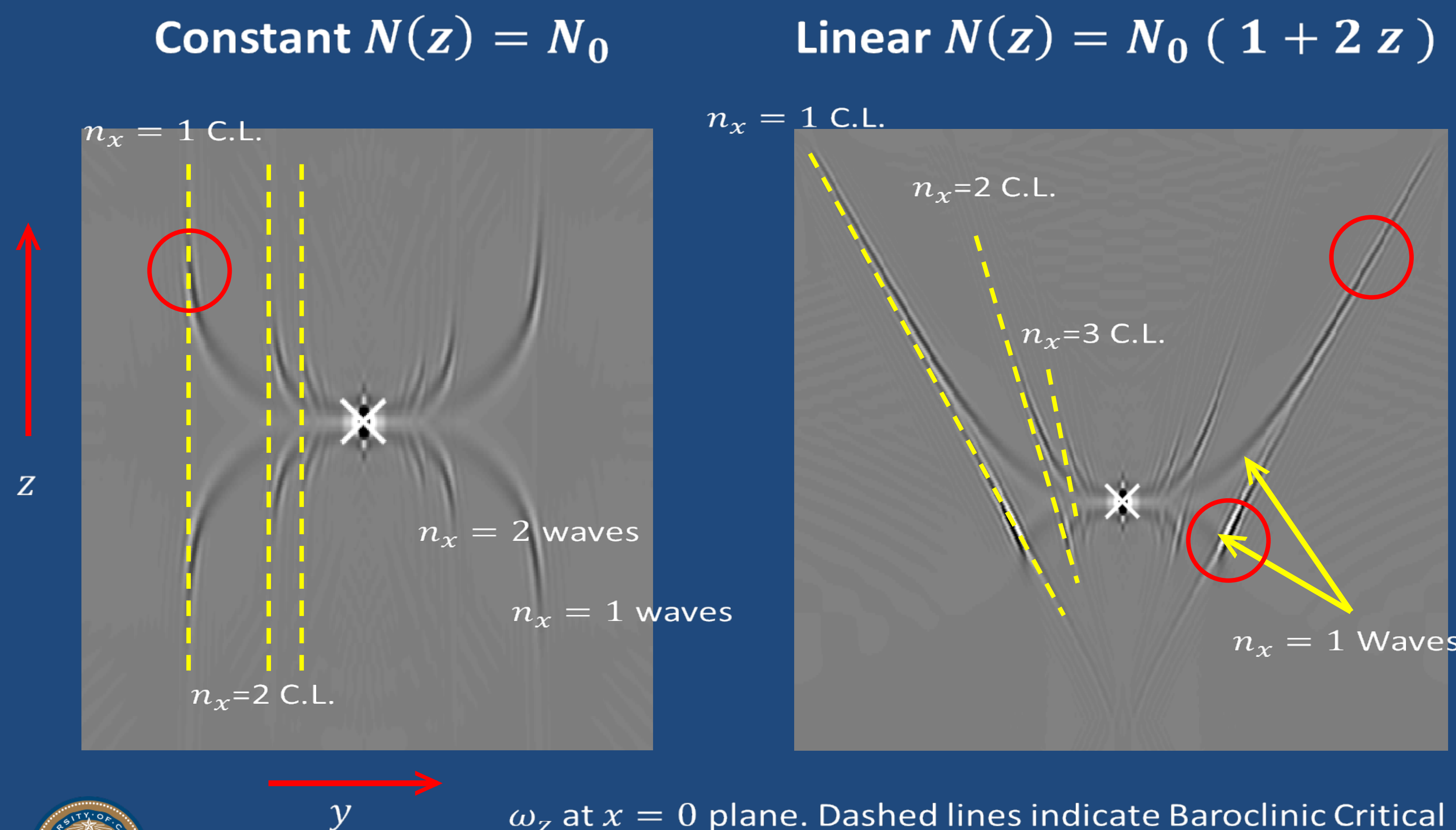
$\sigma = 0.75\, f$

Large shear flow prefers small stream-wise wave numbers

Critical Layer

- Rayleigh equation $\left(\overline{V_x} - c\right)(\phi'' - k_x\,\phi)\cdots$ in unidirectional, dissipation-less shear flows with $\underline{\overline{V}} = \overline{V_x}(y)\,\hat{x}$
- At location y_c where an eigenmode's phase velocity $\mathrm{c} = \omega/k_x = \overline{V_x}(y_c)$, there is a critical layer
- Adding stratification, *baroclinic* critical layer occurs at $y_c = \pm\frac{\omega}{\sigma k_x} \pm \frac{N(z)}{\sigma k_x}$

Wave Packets and Critical Layers



ω_z at $x = 0$ plane. Dashed lines indicate Baroclinic Critical Layers. $\sigma/f = 3/4$ and $f/N_0 = 2/3$

New Small k_x Dispersion Relation WKB Plus Small k_x Approximations

- $k_x \ll k_y \sim k_z$ for large shear σ
- Dispersion Relation:

$$\omega = \omega_0 + \underline{k} \cdot \underline{\overline{V}}$$

- Intrinsic frequency:

$$\omega_0^2 = \frac{N^2(z)\, k_y^2}{k_y^2 + k_z^2} + \frac{f(f - \sigma)\, k_z^2}{k_y^2 + k_z^2}$$

$$\approx N^2(z) \sin^2\theta + f(f - \sigma)\cos^2\theta$$

- Allowed frequency ranges:
  - $f(f - \sigma) \leq \omega_0^2 \leq N^2(z)$
  - $N^2(z) \leq \omega_0^2 \leq f(f - \sigma)$

Cherry Picking Validation: Local Geometric Forms

- Small k_x Dispersion Relation:

$$\omega_0^2(y_l) \approx N^2(z_l)\sin^2\theta + f(f - \sigma)\cos^2\theta$$

$$\left(\frac{dz}{dy}\right)^2 \approx \tan^2\theta \approx \frac{\omega_0^2(y_l) - f(f - \sigma)}{N^2(z_l) - \omega_0^2(y_l)}$$

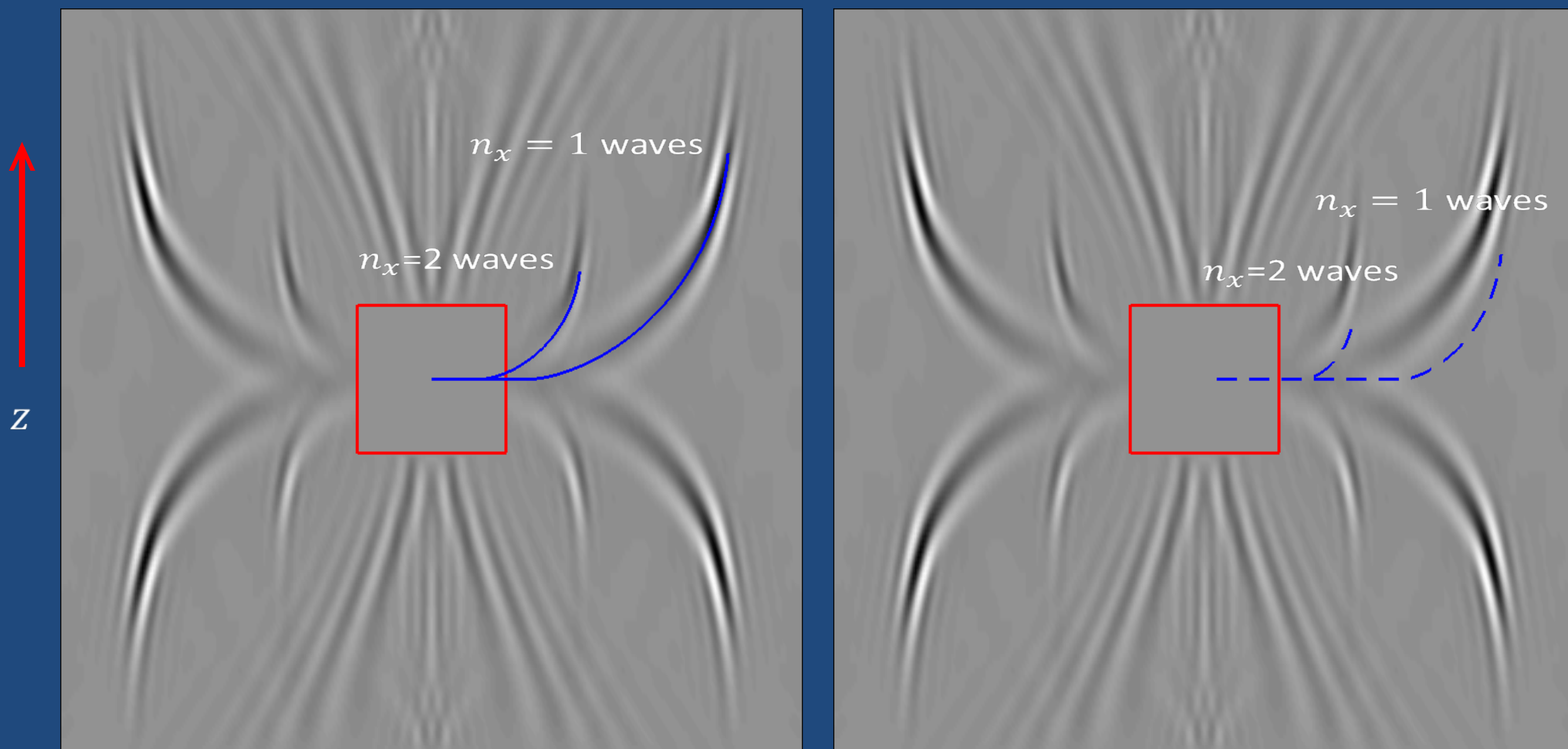
- Textbook dispersion relation:

$$\omega_0^2(y_l) \approx N^2(z_l)\sin^2\theta + f^2\cos^2\theta$$

$$\left(\frac{dz}{dy}\right)^2 \approx \tan^2\theta \approx \frac{\omega_0^2(y_l) - f^2}{N^2(z_l) - \omega_0^2(y_l)}$$

Validation of Small k_x Dispersion Relation
Constant N_0 Case

Small k_x Dispersion Relation | **Textbook Dispersion Relation**



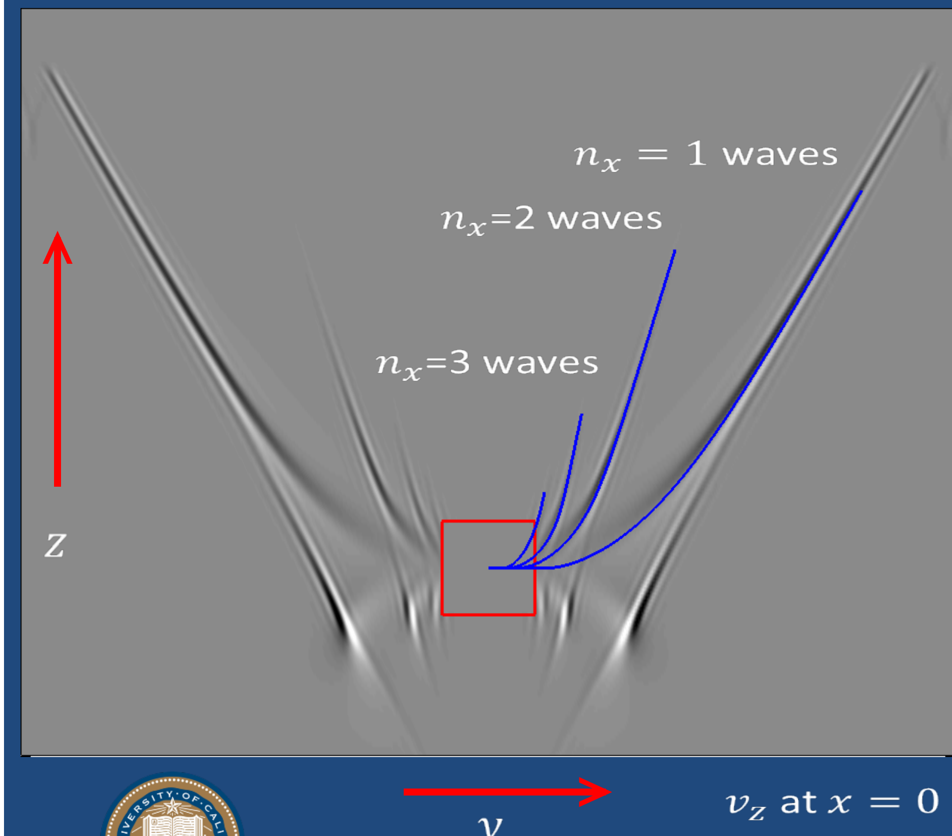
y

v_z at $x = 0$ plane

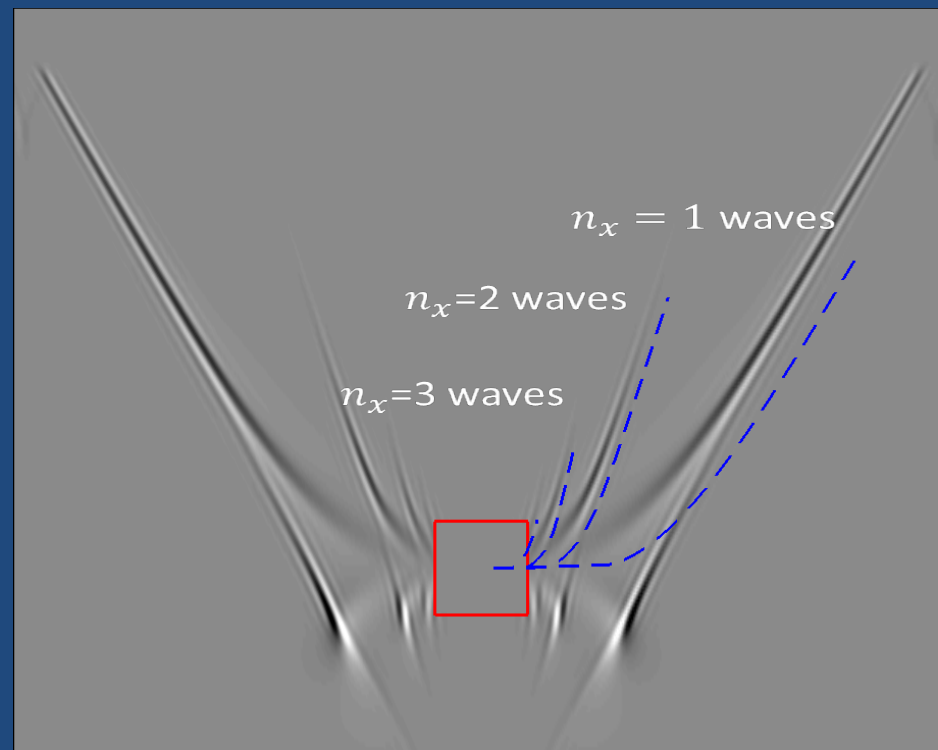
Blue: projection of rays onto $x = 0$ plane

Validation of Small k_x Dispersion Relation Linear $N(z)$ Case

Small k_x Dispersion Relation



Textbook Dispersion Relation



v_z at $x = 0$ plane

Blue: projection of rays onto $x = 0$ plane

Conclusion

- The new small k_x dispersion relation works much better compared to traditional one
- Both horizontal and vertical spacing of the lattice structures determined by locations where waves and critical layers have the same stream-wise wave number (n_x) interact at
- More serious validation of small k_x dispersion relation is ongoing